Math 2940 Quiz 4
Solutions
October 31st, 2019

Name:

NetID:

In $\mathbb{P}_2$, find the change-of-coordinates matrix from the basis $\mathcal{B} = \{1-3t^2,\ 2+t-5t^2,\ 1+2t\}$ to the standard basis $\{1,\ t,\ t^2\}$.

Then write t^2 as a linear combination of the polynomials in $\mathcal{B}$.

Solution:

First, we write the standard basis as $\mathcal{C} = \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$ and the basis $\mathcal{B}$ as $\left\{ \begin{bmatrix}1\\0\\-3\end{bmatrix}, \begin{bmatrix}2\\1\\-5\end{bmatrix}, \begin{bmatrix}1\\2\\0\end{bmatrix} \right\}$.

The change-of-coordinates matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{\mathcal{P}}$ should map the first basis element $\boldsymbol{b}_1$ in basis $\mathcal{B}$ to the same vector $\boldsymbol{b}_1$, but now in basis $\mathcal{C}$. This means:

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{\mathcal{P}} \begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\0\\-3\end{bmatrix}$$

Similarly for $\boldsymbol{b}_2$ and $\boldsymbol{b}_3$,

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{\mathcal{P}} \begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}2\\1\\-5\end{bmatrix}, \qquad \underset{\mathcal{C}\leftarrow\mathcal{B}}{\mathcal{P}} \begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}1\\2\\0\end{bmatrix}$$

Therefore the change-of-coordinates matrix is

$$\boxed{\underset{\mathcal{C}\leftarrow\mathcal{B}}{\mathcal{P}} = \begin{bmatrix}1 & 2 & 1\\0 & 1 & 2\\-3 & -5 & 0\end{bmatrix}}$$

To find a linear combination that adds up to t^2, this means we want to find coefficients a, b, and c such that:

$$a\begin{bmatrix}1\\0\\-3\end{bmatrix} + b\begin{bmatrix}2\\1\\-5\end{bmatrix} + c\begin{bmatrix}1\\2\\0\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

which is equivalent to the augmented matrix

$$\left[\begin{array}{ccc|c}1 & 2 & 1 & 0\\0 & 1 & 2 & 0\\-3 & -5 & 0 & 1\end{array}\right]$$

(continued on next page)

Row reducing,

$$
\begin{aligned}
&\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 3 & 1 \end{array}\right] \\
&\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \\
&\sim \left[\begin{array}{ccc|r} 1 & 2 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right] \\
&\sim \left[\begin{array}{ccc|r} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right]
\end{aligned}
$$

Therefore the linear combination is $\boxed{3(1-3t^2) - 2(2+t-5t^2) + (1+2t).}$

We can check that

$$
\begin{aligned}
&3(1-3t^2) - 2(2+t-5t^2) + (1+2t) \\
&= (3-4+1) + (-2+2)t + (-9+10)t^2 \\
&= t^2
\end{aligned}
$$

as desired.