Math 2940 Worksheet

Rank, Determinants

Week 9

October 24th, 2019

This worksheet covers material from **Sections 3.1-3.3, 4.6**, and **4.7**. Please work in collaboration with your classmates to complete the following exercises - this means sharing ideas and asking each other questions.

Question 1. Mark each statement True or False. If True, provide a brief explanation. If False, provide a counterexample.

(a) Adding a multiple of one row to another row does not affect the determinant of a matrix.

(b) If the columns of A are linearly dependent, then $\det(A) = 0$.

(c) $\det(A + B) = \det(A) + \det(B)$

(d) The determinant of A is the product of the diagonal entries in A.

(e) If $\det(A) = 0$, then two rows or two columns are the same, or a row or a column is zero.

(f) $\det\left(A^{-1}\right) = (-1)\det(A)$

(g) $\det(AB) = \det(BA)$

Question 2. Find the determinant of the following matrix by first using row reduction.

$$\begin{bmatrix} 1 & 5 & -4 \\ -1 & -4 & 5 \\ -2 & -8 & 7 \end{bmatrix}$$

Question 3. Let $\boldsymbol{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} c \\ 0 \end{bmatrix}$, where a, b, and c are positive.

(a) Compute the area of the parallelogram determined by $\mathbf{0}$, $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{u} + \boldsymbol{v}$.

(b) Compute the determinants of $\begin{bmatrix} a & c \\ b & 0 \end{bmatrix}$ and $\begin{bmatrix} c & a \\ 0 & b \end{bmatrix}$. How do these compare to your answers from part (a)?

Question 4. The following two matrices are row equivalent.

$$A = \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ -2 & 3 & -3 & -3 & -4 \\ 4 & -6 & 9 & 5 & 9 \\ -2 & 3 & 3 & -4 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Find rank(A) and dim(Nul(A))

(b) How many pivot columns are in a row echelon form of A^T?

(c) Suppose a new 4×7 matrix C has four pivot columns. Is Col(A) $= \mathbb{R}^4$? Is Nul(A) $= \mathbb{R}^3$? Why or why not?

Question 5. Let T be a linear transformation. Connect each item in the column on the left with all synonymous items in the column on the right:

	dim(domain T)
	dim(codomain T)
$\text{rank}(T)$	
	$\dim(\text{Col}(T))$
$\dim(\text{Nul}(T))$	
	number of linearly independent columns of T
$\text{rank}(T) + \dim(\text{Nul}(T))$	
	number of pivotal columns of T
	number of non-pivotal columns of T

Question 6.

(a) Show that if A is invertible, then $\det(A^{-1}) = \frac{1}{\det(A)}$.

(b) Let A and P be square matrices, with P invertible. Show that $\det(PAP^{-1}) = \det(A)$.

(c) Let U be a square matrix such that $U^T U = I$. Show that $\det(U) = \pm 1$.

Question 7. In this problem, we'll explore how low rank factorizations can be very efficient in terms of storage and computations. In one of the homework problems from the textbook this week, you'll show that a rank-1 matrix can be factorized as the product of two vectors, i.e. $\widetilde{A} = \boldsymbol{u}\boldsymbol{v}^T$ if $\widetilde{A}$ is rank 1.

(a) Suppose $\boldsymbol{u}_1$ is linearly independent from $\boldsymbol{u}_2$, and $\boldsymbol{v}_1$ is linearly independent from $\boldsymbol{v}_2$. What is the rank of the matrix $\overline{A} = [\boldsymbol{u}_1\ \boldsymbol{u}_2][\boldsymbol{v}_1\ \boldsymbol{v}_2]^T$? Can you interpret $\overline{A}$ as a sum of rank-1 matrices?

(b) Suppose that an $m \times n$ matrix A has rank r, with r much smaller than m and n, and $A = UV^T$. If both U and V have linearly independent columns, what are the sizes of the matrices U and V?

(c) How can you store the matrix A to minimize storage costs?

(d) What is the fastest way to compute BA, where B is an $m \times m$ matrix? Count the number of multiplications used.

Answer to Question 1.

(a) True. One way of thinking about this, is that this row operation corresponds to multiplying the original matrix (which I'll call A), with an elementary matrix (which I'll call E). For instance, we could have something like $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & 0 & 1 \end{bmatrix}$. The point is that the elementary matrix E is always either upper or lower triangular, with only ones along the main diagonal. This means that $\det(E) = 1$. So therefore $\det(EA) = \det(E)\det(A) = \det(A)$, and the determinant is unchanged.

Another way of thinking about this is using the fact that the determinant is multi-linear in rows. If our original matrix A has rows $A = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}$, then the row operation would lead to something like $\begin{bmatrix} R_1 \\ R_2 + cR_1 \\ R_3 \end{bmatrix}$. Because the determinant is multi-linear, we have

$$\det\left(\begin{bmatrix} R_1 \\ R_2 + cR_1 \\ R_3 \end{bmatrix}\right) = \det\left(\begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}\right) + \det\left(\begin{bmatrix} R_1 \\ cR_1 \\ R_3 \end{bmatrix}\right)$$

That last matrix is clearly noninvertible, because when row reducing we will get a row of all zeros. Therefore it's determinant is zero, and we get

$$\det\left(\begin{bmatrix} R_1 \\ R_2 + cR_1 \\ R_3 \end{bmatrix}\right) = \det\left(\begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}\right)$$

This is true no matter how many rows there are, or which row is added to the other.

(b) True. If the columns of A are linearly dependent, then A is a noninvertible matrix, and therefore $\det(A) = 0$.

(c) False. For a counterexample, consider

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

then

$$\det(A+B) = \det\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) = 0, \qquad \det(A) = 1, \qquad \det(B) = 1$$

so $\det(A+B) \neq \det(A) + \det(B)$.

(d) False. This is true when A is a triangular mtarix, but not in general. For a counterexample, consider

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

The product of the diagonal entries is 0, but A is invertible because you just need to swap rows to get to the identity matrix.

(e) $\boxed{\text{False.}}$ There are actually many more ways of being linearly dependent then just those conditions. For example, consider

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

(f) $\boxed{\text{False.}}$ This is not true in general. For a specific counterexample, consider the identity matrix:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$\det(A) = 1, \qquad \det(A^{-1}) = 1 \neq -\det(A)$$

In general, if we have an invertible matrix A, then $A^{-1}A = I$, so

$$\begin{aligned} \det(A^{-1}A) &= \det(I) \\ \det(A^{-1})\det(A) &= 1 \\ \det(A^{-1}) &= \frac{1}{\det(A)} \end{aligned}$$

(g) If A and B are square matrices (which I forgot to include as part of the question), then this is $\boxed{\text{True.}}$ Even though AB and BA are usually different matrices, they actually have the same determinant because:

$$\det(AB) = \det(A)\det(B) = \det(B)\det(A) = \det(BA)$$

However, this is not always true if A and B are not square matrices. Consider:

$$A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \end{bmatrix}$$
$$\det(AB) = \det\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = 0, \qquad \det(BA) = \det\left(\begin{bmatrix} 1 \end{bmatrix}\right) = 1$$

Answer to Question 2. First we row reduce the matrix to echelon form:

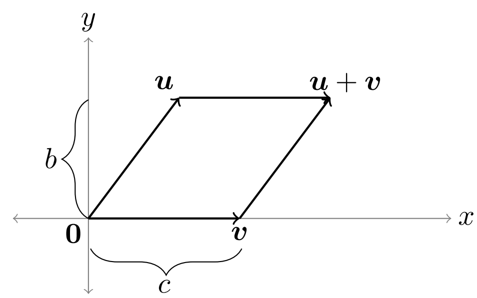
$$\begin{aligned} &\begin{bmatrix} 1 & 5 & -4 \\ -1 & -4 & 5 \\ -2 & -8 & 7 \end{bmatrix} \begin{matrix} \\ \leftarrow R_2 + R_1 \\ \\ \end{matrix} \\ \sim &\begin{bmatrix} 1 & 5 & -4 \\ 0 & 1 & 1 \\ -2 & -8 & 7 \end{bmatrix} \begin{matrix} \\ \\ \leftarrow R_3 + (2) \cdot R_1 \end{matrix} \\ \sim &\begin{bmatrix} 1 & 5 & -4 \\ 0 & 1 & 1 \\ 0 & 2 & -1 \end{bmatrix} \begin{matrix} \\ \\ \leftarrow R_3 - (2) \cdot R_2 \end{matrix} \\ \sim &\begin{bmatrix} 1 & 5 & -4 \\ 0 & 1 & 1 \\ 0 & 0 & -3 \end{bmatrix} \end{aligned}$$

Because we only used row replacement operations (no swaps or scaling), the determinant of the matrix did not change. Since the final matrix is triangular, we get the determinant by multiplying the diagonal dentries, so

$$\boxed{\det = -3}$$

Answer to Question 3.

(a) The parallelogram should look something like this:



Since the base of the parallelogram is c and the height is b, the area is:

$$\boxed{\text{Area } = (\text{base}) \cdot (\text{height}) = bc}$$

(b) Since these are only 2×2 matrices, we can directly compute the determinants as:

$$\boxed{\det\left(\begin{bmatrix} a & c \\ b & 0 \end{bmatrix}\right) = -bc, \qquad \det\left(\begin{bmatrix} c & a \\ 0 & b \end{bmatrix}\right) = bc}$$

Compared to our previous answers, we see that:

$$\text{Area} = \left|\det\left(\begin{bmatrix} \boldsymbol{u} & \boldsymbol{v} \end{bmatrix}\right)\right|$$

Answer to Question 4.

(a) Since the two matrices are row equivalent, we know that $\text{rank}(A) = \text{rank}(B)$, and $\dim(\text{Nul}(A)) = \dim(\text{Nul}(B))$.

We can easily see that B has three pivots, therefore $\boxed{\text{rank}(A) = 3}$

And since B has two free variables, we see that $\boxed{\dim(\text{Nul}(B)) = 2}$

(b) First we recall that the rank of a matrix is equal to the number of pivot columns.

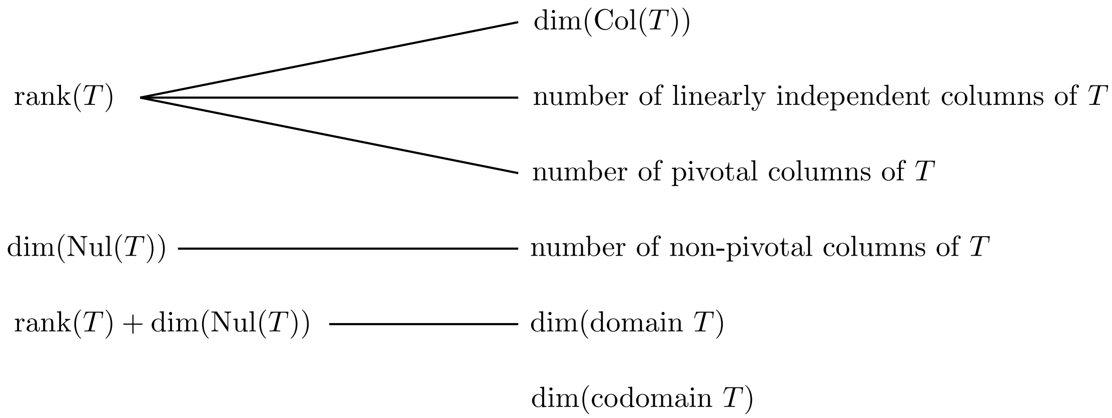
Since $\text{rank}(A) = \text{rank}(A^T) = 3$, we see that $\boxed{A^T \text{ has 3 pivot columns}}$

(c) Since C has four pivot columns, $\dim(\text{Col}(A)) = 4$. Since $\text{Col}(A)$ is a subspace of $\mathbb{R}^4$ and the only 4-dimensional subspace of $\mathbb{R}^4$ is $\mathbb{R}^4$ itself, it follows that $\boxed{\text{Col}(A) = \mathbb{R}^4}$

Because C has three free variables, it follows that $\dim(\text{Nul}(A)) = 3$. However, this does not mean that $\text{Nul(A)} = \mathbb{R}^3$.

In fact, the nullspace is a subspace of the domain. Because C has 7 rows, we know that $\text{Nul}(A)$ is a subspace of $\mathbb{R}^7$. So in fact $\boxed{\text{Nul}(A) \neq \mathbb{R}^3}$, because $\text{Nul}(A)$ is a set that lives inside of $\mathbb{R}^7$, not $\mathbb{R}^3$. If we wanted to find a basis for $\text{Nul}(A)$, the basis elements would actually be vectors with seven entries, not three.

Answer to Question 5. Rearranging the columns, there should be connections:



Answer to Question 6.

(a) If A is invertible, then $A^{-1}A = I$, where I is the identity matrix. Taking the determinant of both sides,

$$\det(A^{-1}A) = \det(I)$$
$$\det(A^{-1})\det(A) = 1$$

$$\boxed{\det(A^{-1}) = \frac{1}{\det(A)}}$$

(b) We can split up the derivative as follows

$$\det(PAP^{-1}) = \det(P)\det(A)\det(P^{-1})$$

using the result from the last part,

$$\boxed{\det(PAP^{-1}) = \det(P)\det(A)\frac{1}{\det(P)} = \det(A)}$$

(c) We first take the determinant of both sides to get

$$\det(U^T U) = \det(I)$$
$$\det(U^T)\det(U) = 1$$

Since $\det(U^T) = \det(U)$, we get

$$\det(U)^2 = 1$$

and solving for $\det(U)$,

$$\boxed{\det(U) = \pm 1}$$

Answer to Question 7.

(a) To figure out the rank of the matrix $\overline{A}$, we can look at $\overline{A}$ as a multiplication of the form

$$\overline{A} = \begin{bmatrix} \\ \boldsymbol{u}_1 & \boldsymbol{u}_2 \\ \\ \end{bmatrix} \begin{bmatrix} & \boldsymbol{v}_1 & \\ & \boldsymbol{v}_2 & \end{bmatrix}$$

We can see that every single row of A is some linear combination of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. This means that we can use row operations to reduce $\overline{A}$ to:

$$\overline{A} \sim \begin{bmatrix} & \boldsymbol{v}_1 & \\ & \boldsymbol{v}_2 & \\ & \mathbf{0} & \\ & \mathbf{0} & \\ & \vdots & \end{bmatrix}$$

Since $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are linearly independent, this means that $\boxed{\text{rank}(\overline{A}) = 2.}$

Moreover, when we look at the matrix multiplication above, we can see that each entry of A is the sum of two terms, one coming from $\boldsymbol{u}_1\boldsymbol{v}_1^T$, and one coming from $\boldsymbol{u}_2\boldsymbol{v}_2^T$. Therefore we can see $\overline{A}$ as a sum of two rank-1 matrices:

$$\boxed{\overline{A} = \boldsymbol{u}_1\boldsymbol{v}_1^T + \boldsymbol{u}_2\boldsymbol{v}_2^T}$$

(b) In order for the matrix multiplication UV^T to result in a $m \times n$ matrix A, there must be some integer k such that U is an $m \times k$ matrix, and V is an $n \times k$ matrix.

Using similar logic as part (a), we see that each row of UV^T is a linear combination of k linearly independent rows. In order for UV^T to have rank r, we then need $k = r$. Therefore

$$\boxed{U \text{ is an } m \times r \text{ matrix,} \quad \text{and} \quad V \text{ is an } n \times r \text{ matrix}}$$

(c) If we know that $A = UV^T$, then there are two possible ways of storing A:

- Store A directly, which requires storing mn entries
- Store U and V, which requires $mr + nr = (m + n)r$ entries.

In the case where r is much smaller than m and n, we see that storing U and V^T is much more efficient than storing A directly.

(d) Here are two possible ways of computing BA:

- Compute BA directly. Since there are mn entries, and calculating each entry requires m multiplications, this takes m^2n multiplications overall
- Compute $(BU)V^T$. The first multiplication BU requires computing mr entries, where each entry requires m multiplications. The second multiplication with V^T then requires computing mn entries, where each entry requires r multiplications. In total this requires $m^2r + mnr = (m + n)mr$ multiplications.

In the case where r is much smaller than m and n, we also see that computing $(BU)V^T$ is more efficient than computing BA directly.