Math 2940 Worksheet

Coordinate Systems, Dimension

Week 8

October 17th, 2019

This worksheet covers material from **Sections 4.4 and 4.5**. Please work in collaboration with your classmates to complete the following exercises - this means sharing ideas and asking each other questions.

Question 1. The set $\mathcal{B} = \{1+t,\ 1+t^2,\ t+t^2\}$ is a basis for $\mathbb{P}_2$ (the vector space of all polynomials with degree at most 2). Find the coordinate vector of $\boldsymbol{p}(t) = 6 + 3t - t^2$ relative to $\mathcal{B}$.

Question 2. Determine the dimensions of the nullspace and the column space for the following matrices:

$$A = \begin{bmatrix} 1 & 0 & 9 & 5 \\ 0 & 0 & 1 & -4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -6 & 9 & 0 & -2 \\ 0 & 1 & 2 & -4 & 5 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Question 3. For this question, V is a nonzero finite-dimensional vector space. Mark each statement True or False, and justify each answer.

(a) The number of pivot columns of a matrix equals the dimension of its column space.

(b) A plane in $\mathbb{R}^3$ is a two-dimensional subspace of $\mathbb{R}^3$

(c) The dimension of the vector space $\mathbb{P}_4$ is 4.

(d) If a set $\{\boldsymbol{v}_1, ...\boldsymbol{v}_p\}$ spans a finite-dimensional vector space V and if T is a set of more than p vectors in V, then T is linearly dependent.

(e) The only three-dimensional subspace of $\mathbb{R}^3$ is $\mathbb{R}^3$ itself.

(f) If there exists a linearly dependent set $\{\boldsymbol{v}_1, ...\boldsymbol{v}_p\}$ in V, then dim $V \leq p$.

(g) If every set of p elements in V fails to span V, then dim $V > p$.

Question 4.

(a) Determine the degree 4 polynomial that interpolates the data $(0, 1)$, $(1, 4)$, $(2, 9)$, $(3, 16)$, and $(4, 25)$ by computing the coordinates of the interpolant in the monomial basis, i.e. in the basis $\{1, x, x^2, x^3, x^4\}$.

Homework 6 introduces the Chebyshev polynomials

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_2(x) = 2x^2 - 1 \quad T_3(x) = 4x^3 - 3x, \quad T_4(x) = 8x^4 - 8x^2 + 1,$$

as a basis for $\mathbb{P}_4$, the polynomials of degree ≤ 4.

(b) Using a change of coordinates from the monomial basis and your answer from part (a), compute the coordinates of the interpolant in the Chebyshev basis.

Answer to Question 1. To find the coordinate vector, we want to find a linear combination of the elements of $\mathcal{B}$ that equals $\boldsymbol{p}(t)$.
More specifically, we want to find coefficients a, b, and c such that:

$$a(1+t)+b(1+t^2)+c(t+t^2)=6+3t-t^2$$

Comparing like terms, we can turn these into vectors and get three linear equations in three unknowns:

$$a\begin{bmatrix}1\\1\\0\end{bmatrix}+b\begin{bmatrix}1\\0\\1\end{bmatrix}+c\begin{bmatrix}0\\1\\1\end{bmatrix}=\begin{bmatrix}6\\3\\-1\end{bmatrix}$$

Turning this into an augmented matrix and row reducing,

$$\left[\begin{array}{ccc|c}1&1&0&6\\1&0&1&3\\0&1&1&-1\end{array}\right]$$
$$\sim\left[\begin{array}{ccc|c}1&1&0&6\\0&-1&1&-3\\0&1&1&-1\end{array}\right]$$
$$\sim\left[\begin{array}{ccc|c}1&1&0&6\\0&-1&1&-3\\0&0&2&-4\end{array}\right]$$
$$\sim\left[\begin{array}{ccc|c}1&1&0&6\\0&-1&1&-3\\0&0&1&-2\end{array}\right]$$
$$\sim\left[\begin{array}{ccc|c}1&1&0&6\\0&1&0&1\\0&0&1&-2\end{array}\right]$$
$$\sim\left[\begin{array}{ccc|c}1&0&0&5\\0&1&0&1\\0&0&1&-2\end{array}\right]$$

So the coordinate vector is $\boxed{\begin{bmatrix}5\\1\\-2\end{bmatrix}}$.

Answer to Question 2. The dimension of Nul A will be the number of free variables, since each free variable will correspond to a basis of the nullspace.

The dimension of Col A will be the number of linearly independent columns, which is equal to the number of pivot columns in echelon form.

Applying this to the two examples,

$$A=\begin{bmatrix}1&0&9&5\\0&0&1&-4\end{bmatrix}$$

Here we can see that there are two pivots and two free variables, so $\boxed{\dim(\text{Col } A\ )=2}$ and $\boxed{\dim(\text{Nul } A\ )=2}$.

For

$$B = \begin{bmatrix} 1 & -6 & 9 & 0 & 2 \\ 0 & 1 & 2 & -4 & 5 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We see that we have three pivots and two free variables, so $\boxed{\dim(\text{Col } A\ ) = 3}$ and $\boxed{\dim(\text{Nul } A\ ) = 2}$.

Answer to Question 3.

(a) $\boxed{\text{True.}}$ The pivot columns are linearly independent and span the column space, so they form a basis for the column space. Thus the number of pivot columns equals the dimension of the column space.

(b) $\boxed{\text{False.}}$ Not all planes are actually subspaces, because not all planes go through the origin (for example, the plane $z = 1$). However, planes that go through the origin are two-dimensional subspaces of $\mathbb{R}^3$.

(c) $\boxed{\text{False.}}$ The 4 in $\mathbb{P}_4$ stands for polynomials of *degree* 4, not *dimension 4*. To describe polynomials of degree 4, we actually need 5 coefficients:

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$$

so $\dim(\mathbb{P}_4) = 5$.

(d) $\boxed{\text{True.}}$ Given that the set $\{\boldsymbol{v}_1, ..., \boldsymbol{v}_p\}$ spans V, this tells us that $\dim(V) \leq p$. Therefore every set of more than p vectors in V is linearly independent.

(e) $\boxed{\text{True.}}$ Any three-dimensional subspace of $\mathbb{R}^3$ must have a corresponding basis of 3 linearly independent vectors. These three linearly independent vectors must span all of $\mathbb{R}^3$, so the subspace has to be $\mathbb{R}^3$.

(f) $\boxed{\text{False.}}$ It's very easy to come up with examples of small linearly dependent sets in high-dimensional spaces. For example, $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \right\}$ is a linearly dependent set of two elements in $\mathbb{R}^3$.

(g) $\boxed{\text{True.}}$ If every set of p elements does not span V, then that means every basis of V must have more than p elements. Therefore $\dim V > p$.

Answer to Question 4.

(a) To interpolate this data, we want to find the degree 4 polynomial that passes through all 5 of those points. This will give us a linear system of 5 equations (the 5 points we need to pass through) in 5 variables (the 5 coefficients).

Writing the interpolating polynomial as

$$a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$$

We plug in our points to get the system of 5 equations:

$$\begin{aligned}
a_0 \qquad\qquad\qquad\qquad\qquad\qquad &= 1 \\
a_0 + a_1 + a_2 + a_3 + a_4 &= 4 \\
a_0 + 2a_1 + 4a_2 + 8a_3 + 16a_4 &= 9 \\
a_0 + 3a_1 + 9a_2 + 27a_3 + 81a_4 &= 16 \\
a_0 + 4a_1 + 16a_2 + 64a_3 + 256a_4 &= 25
\end{aligned}$$

which we can turn into the following augmented matrix:

$$\left[\begin{array}{ccccc|c}
1 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 4 \\
1 & 2 & 4 & 8 & 16 & 9 \\
1 & 3 & 9 & 27 & 81 & 16 \\
1 & 4 & 16 & 64 & 256 & 25
\end{array}\right]$$

which we can row reduce the matrix to echelon form as follows:

$$\sim \left[\begin{array}{ccccc|c}
1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & 1 & 1 & 3 \\
0 & 2 & 4 & 8 & 16 & 8 \\
0 & 3 & 9 & 27 & 81 & 15 \\
0 & 4 & 16 & 64 & 256 & 24
\end{array}\right]$$

$$\sim \left[\begin{array}{ccccc|c}
1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & 1 & 1 & 3 \\
0 & 0 & 2 & 6 & 14 & 2 \\
0 & 0 & 6 & 24 & 78 & 6 \\
0 & 0 & 12 & 60 & 252 & 12
\end{array}\right]$$

$$\sim \left[\begin{array}{ccccc|c}
1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & 1 & 1 & 3 \\
0 & 0 & 2 & 6 & 14 & 2 \\
0 & 0 & 0 & 6 & 36 & 0 \\
0 & 0 & 0 & 24 & 168 & -0
\end{array}\right]$$

$$\sim \left[\begin{array}{ccccc|c}
1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & 1 & 1 & 3 \\
0 & 0 & 2 & 6 & 14 & 2 \\
0 & 0 & 0 & 6 & 36 & 0 \\
0 & 0 & 0 & 0 & 22 & 0
\end{array}\right]$$

Solving the rest by back substitution,

$$\begin{aligned} 22a_4 &= 0 &\implies&& a_4 &= 0 \\ 6a_3 + 36(0) &= 0 &\implies&& a_3 &= 0 \\ 2a_2 + 6(0) + 14(0) &= 2 &\implies&& a_2 &= 1 \\ a_1 + (1) + (0) + (0) &= 3 &\implies&& a_1 &= 2 \\ a_0 &= 1 \end{aligned}$$

Therefore the interpolating polynomial is:

$$\boxed{y = 1 + 2x + x_2 = (x+1)^2}$$

(b) First, to get a change of coordinates matrix T from the Chebyshev basis to the standard basis, we write the Chebyshev polynomials in terms of the standard monomial basis, and use them as the columns of our matrix T.

$$T = \begin{bmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 2 & 0 & -8 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 8 \end{bmatrix}$$

Now, to go from the monomial basis to the Chebyshev basis, we could compute the inverse matrix T^{-1}.

But if you only want to change the coordinates of a single vector, it's more efficient to solve the linear system $T\boldsymbol{x} = \boldsymbol{b}$, where $\boldsymbol{b}$ is our answer from part (a) in the monomial basis, and $\boldsymbol{x}$ will be our answer in the Chebyshev basis.

We can write this down as the following augmented matrix:

$$\left[\begin{array}{ccccc|c} 1 & 0 & -1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -3 & 0 & 2 \\ 0 & 0 & 2 & 0 & -8 & 1 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8 & 0 \end{array}\right]$$

Which we can solve using back-substitution:

$$\begin{aligned} 8c_4 &= 0 &\implies&& c_4 &= 0 \\ 4c_3 &= 0 &\implies&& c_3 &= 0 \\ 2c_2 + -8(0) &= 1 &\implies&& c_2 &= \frac{1}{2} \\ c_1 + -3(0) &= 2 &\implies&& c_1 &= 2 \\ c_0 - \left(\frac{1}{2}\right) + 1(0) &= 1 &\implies&& c_0 &= \frac{3}{2} \end{aligned}$$

So the coordinates are: $\boxed{\begin{bmatrix} \frac{3}{2} \\ 2 \\ \frac{1}{2} \\ 0 \\ 0 \end{bmatrix}}$, or in other terms,

$$\boxed{(x+1)^2 = \frac{3}{2}T_0(x) + 2T_1(x) + \frac{1}{2}T_2(x)}$$