Math 2940 Worksheet

Factorizations and Vector Spaces

Week 6

October 3rd, 2019

This worksheet covers material from **Sections 2.5, 4.1, and 4.2**. Please work in collaboration with your classmates to complete the following exercises - this means sharing ideas and asking each other questions.

Question 1. Find an LU factorization of

$$A = \begin{bmatrix} -5 & 3 & 4 \\ 10 & -8 & -9 \\ 15 & 1 & 2 \end{bmatrix}$$

Question 2. Show that the set H of all points in $\mathbb{R}^2$ of the form $(3s, 2+5s)$ is not a vector space, by showing that it is not closed under scalar multiplication.

(Find a specific vector $\boldsymbol{u}$ in H and a scalar c such that $c\boldsymbol{u}$ is not in H.)

Question 3. Let $A = \begin{bmatrix} 7 & -3 & 5 \\ -4 & 1 & -5 \\ -5 & 2 & -4 \end{bmatrix}$, $\boldsymbol{v} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$, and $\boldsymbol{w} = \begin{bmatrix} 7 \\ 6 \\ -3 \end{bmatrix}$. Suppose you know that the equations $A\boldsymbol{x} = \boldsymbol{v}$ and $A\boldsymbol{x} = \boldsymbol{w}$ are both consistent. What can you say about the equation $A\boldsymbol{x} = \boldsymbol{v} + \boldsymbol{w}$?

Question 4. Suppose a 3×3 matrix A admits a factorization as $A = PDP^{-1}$, where P is an invertible 3×3 matrix, and D is the diagonal matrix

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$$

Show that this factorization is useful when computing high powers of A. Find fairly simple formulas for A^2, A^3, and A^k (k is a positive integer), using P and the entries in D.

Question 5. The set $\mathbb{P}_3$ of all polynomials of degree less than or equal to 3 is a vector space. We can represent its elements in terms of the following standard basis vectors:

$$1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad x = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \qquad x^2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \qquad x^3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

For example, the polynomial $1 + 2x^2 - x^3$ would correspond to the vector $\begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix}$.

(a) The derivative can be thought of as a linear transformation $D : \mathbb{P}_3 \to \mathbb{P}_3$. What is the matrix corresponding to this linear transformation?

(*Hint*: Think about what this transformation does to each of the basis vectors)

(b) Is the matrix you found in part (a) invertible? Is that the answer you expected from calculus?

Question 6.

(a) The set of *orthogonal* matrices are the set of square matrices Q that satisfy $Q^TQ = I$. Show that the Givens rotation matrix from homework 3 is orthogonal.

$$Q_1 = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Just as elementary row operations lead to the LU factorization, Givens rotations lead to another type of factorization. The *QR factorization* of an $m \times n$ matrix A has the form $A = QR$, where R is an upper triangular matrix and Q is orthogonal.

(b) Suppose R is an invertible $n \times n$ matrix. Show that R^T is also invertible. What is its inverse?

(c) In homework 4, you showed that the *least-squares* solution of an overdetermined system (more equations than unknowns) $A\boldsymbol{x} = \boldsymbol{b}$ satisfies the so-called *normal equations* $A^T A\boldsymbol{x} = A^T\boldsymbol{b}$.

Suppose that $A = QR$ is a QR factorization of A, where R is invertible. Solve the normal equations for $\boldsymbol{x}$ using the QR factorization of A.

Question 7. Find an LU factorization of

$$A = \begin{bmatrix} 2 & -4 & -2 & 3 \\ 6 & -9 & -5 & 8 \\ 2 & -7 & -3 & 9 \\ 4 & -2 & -2 & -1 \\ -6 & 3 & 3 & 4 \end{bmatrix}$$

Answer to Question 1. To compute the LU factorization, we need to reduce the matrix to echelon form.
I'm going to write this solution in two columns, with the right column showing the results of the row operations (which will become U), and the left column keeping track of the actual row operations (which will become L).
The entries in L will come from normalizing the highlighted entries on the right.

$$L = \begin{bmatrix} 1 & & \\ -2 & & \\ -3 & & \end{bmatrix} \qquad A = \begin{bmatrix} -5 & 3 & 4 \\ 10 & -8 & -9 \\ 15 & 1 & 2 \end{bmatrix}$$

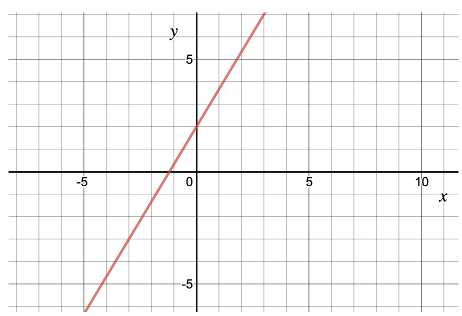
$$L = \begin{bmatrix} 1 & 0 & \\ -2 & 1 & \\ -3 & -5 & \end{bmatrix} \qquad \begin{bmatrix} -5 & 3 & 4 \\ 0 & -2 & -1 \\ 0 & 10 & 14 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & -5 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} -5 & 3 & 4 \\ 0 & -2 & -1 \\ 0 & 0 & 9 \end{bmatrix}$$

So the LU factorization is

$$\boxed{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & -5 & 1 \end{bmatrix} \begin{bmatrix} -5 & 3 & 4 \\ 0 & -2 & -1 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} -5 & 3 & 4 \\ 10 & -8 & -9 \\ 15 & 1 & 2 \end{bmatrix}}$$

Answer to Question 2. If we graph this problem, we can see that H is a line in $\mathbb{R}^2$, but does not actually go through the origin. Graphically, it looks like this:



(If H was a line that went through the origin, it *would* be a vector space.)
But since it doesn't go through the origin, I can first find some vector $\boldsymbol{u}$ in H, say:

$$s = 1 \qquad \Longrightarrow \qquad \boldsymbol{u} = \begin{bmatrix} 3s \\ 2+5s \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

Then if I multiply by $c = 0$, I get $c\boldsymbol{u} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, which is not in H, because

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ is not of the form } \begin{bmatrix} 3s \\ 2+5s \end{bmatrix}$$

$\boxed{\text{Since } H \text{ is not closed under scalar multiplication, it is not a vector space.}}$

Answer to Question 3. Since $A\boldsymbol{x} = \boldsymbol{v}$ and $A\boldsymbol{x} = \boldsymbol{w}$ are both consistent, that means that both $\boldsymbol{v}$ and $\boldsymbol{v}$ are in the column space of A (this is the vector space spanned by the columns of A, often written $\text{Col}(A)$).

Because $\text{Col}(A)$ is a vector space, it is closed under addition. This means that $\boldsymbol{v} + \boldsymbol{w}$ is in $\text{Col}(A)$.

$$\boxed{\text{Therefore, } A\boldsymbol{x} = \boldsymbol{v} + \boldsymbol{w} \text{ is consistent}}$$

Answer to Question 4. If we have that A factors as $A = PDP^{-1}$, then it's much easier to compute high powers of A. For instance,

$$A^2 = A \cdot A = PDP^{-1}PDP^{-1}$$

The $P^{-1}P$ in the middle cancels out, so this becomes

$$A^2 = PD^2P^{-1}$$

Since D is a diagonal matrix, squaring it is very easy: we just square each entry.

$$\boxed{A^2 = P \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/9 \end{bmatrix} P^{-1}}$$

Similarly for A^3:

$$A^3 = A \cdot A \cdot A = PDP^{-1}PDP^{-1}PDP^{-1} = PD^3P^{-1}$$

so

$$\boxed{A^3 = P \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/8 & 0 \\ 0 & 0 & 1/27 \end{bmatrix} P^{-1}}$$

Extrapolating this pattern, for any positive integer k we have

$$\boxed{A^k = P \begin{bmatrix} 1 & 0 & 0 \\ 0 & (1/2)^k & 0 \\ 0 & 0 & (1/3)^k \end{bmatrix} P^{-1}}$$

Answer to Question 5.

(a) To get the matrix, we will find its columns by applying the linear transformation to each standard basis vector:

$$D\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right) \to \frac{d}{dx}(1) = 0 \to \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$D\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right) \to \frac{d}{dx}(x) = 1 \to \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$D\left(\begin{bmatrix}0\\0\\1\\0\end{bmatrix}\right) \to \frac{d}{dx}(x^2) = 2x \to \begin{bmatrix}0\\2\\0\\0\end{bmatrix}$$

$$D\left(\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right) \to \frac{d}{dx}(x^3) = 3x^2 \to \begin{bmatrix}0\\0\\3\\0\end{bmatrix}$$

Using these as our columns, the matrix is:

$$\boxed{\begin{bmatrix}0 & 1 & 0 & 0\\0 & 0 & 2 & 0\\0 & 0 & 0 & 3\\0 & 0 & 0 & 0\end{bmatrix}}$$

(b) The matrix is $\boxed{\text{not invertible}}$, because the first column is all zeros, so the columns are not linearly independent.

This is actually what we should expect from calculus, since we know there are infinitely many functions with the same derivative (that's why we need to add the $+C$ when taking an anti-derivative). That means the derivative is not one-to-one, so it shouldn't be invertible.

Answer to Question 6.

(a) We can check this one directly by computing $Q_1^TQ_1$:

$$Q_1^TQ_1 = \begin{bmatrix}\cos(\theta) & \sin(\theta) & 0\\-\sin(\theta) & \cos(\theta) & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}\cos(\theta) & -\sin(\theta) & 0\\\sin(\theta) & \cos(\theta) & 0\\0 & 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}\cos^2(\theta)+\sin^2(\theta) & \sin(\theta)\cos(\theta)-\sin(\theta)\cos(\theta) & 0\\\sin(\theta)\cos(\theta)-\sin(\theta)\cos(\theta) & \cos^2(\theta)+\sin^2(\theta) & 0\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

so the Givens rotation matrix Q_1 is orthogonal.

(b) Since our matrix R is invertible, that means there is a matrix R such that:

$$R^{-1}R = I$$

Taking the transpose of both sides,

$$(R^{-1}R)^T = I^T = I$$

for the left hand side, we can use the rule that $(AB)^T = B^TA^T$ to get:

$$R^T\left(R^{-1}\right)^T = I$$

We can also do the same thing starting from

$$RR^{-1} = I$$

to get

$$\left(R^{-1}\right)^T R^T = I$$

$$\boxed{\text{This means that } R^T \text{ is invertible, and its inverse is } \left(R^{-1}\right)^T}$$

(c) The normal equations are:

$$A^T A\boldsymbol{x} = A^T\boldsymbol{b}$$

Using the QR factorization $A = QR$,

$$(QR)^T(QR)\boldsymbol{x} = (QR)^T\boldsymbol{b}$$

$$R^T Q^T QR\boldsymbol{x} = R^T Q^T\boldsymbol{b}$$

because Q is orthogonal, the Q^TQ in the left hand side cancels, and

$$R^T R\boldsymbol{x} = R^T Q^T\boldsymbol{b}$$

Since R is invertible, R^T is invertible, so we can multiply both sides on the left by $\left(R^T\right)^{-1}$ to get

$$R\boldsymbol{x} = Q^T\boldsymbol{b}$$

Multiplying both sides on the left by R^{-1},

$$\boxed{\boldsymbol{x} = R^{-1}Q^T\boldsymbol{b}}$$

(In practice, we wouldn't actually compute R^{-1}. To solve this efficiently, we would actually compute $Q^T\boldsymbol{b}$, and then use back-substitution to solve $R\boldsymbol{x} = Q^T\boldsymbol{b}$, since R is already in echelon form.)

Answer to Question 7. We can compute the LU factorization in the two-column way I used for Question 1 as follows:

$$L = \begin{bmatrix} 1 & & & & \\ 3 & & & & \\ 1 & & & & \\ 2 & & & & \\ -3 & & & & \end{bmatrix} \qquad A = \begin{bmatrix} 2 & -4 & -2 & 3 \\ 6 & -9 & -5 & 8 \\ 2 & -7 & -3 & 9 \\ 4 & -2 & -2 & -1 \\ -6 & 3 & 3 & 4 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & & & \\ 3 & 1 & & & \\ 1 & -1 & & & \\ 2 & 2 & & & \\ -3 & -3 & & & \end{bmatrix} \qquad \begin{bmatrix} 2 & -4 & -2 & 3 \\ 0 & 3 & 1 & -1 \\ 0 & -3 & -1 & 6 \\ 0 & 6 & 2 & -7 \\ 0 & -9 & -3 & 13 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 & & \\ 3 & 1 & 0 & & \\ 1 & -1 & 1 & & \\ 2 & 2 & -1 & & \\ -3 & -3 & 2 & & \end{bmatrix} \qquad \begin{bmatrix} 2 & -4 & -2 & 3 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 2 & 2 & -1 & 1 & 0 \\ -3 & -3 & 2 & 0 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 2 & -4 & -2 & 3 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So the LU factorization is:

$$\boxed{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 2 & 2 & -1 & 1 & 0 \\ -3 & -3 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -4 & -2 & 3 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -4 & -2 & 3 \\ 6 & -9 & -5 & 8 \\ 2 & -7 & -3 & 9 \\ 4 & -2 & -2 & -1 \\ -6 & 3 & 3 & 4 \end{bmatrix}}$$