Math 2940 Worksheet

Linear Transformations

Week 4

September 19th, 2019

This worksheet covers material from **Section 1.7-1.9**. Please work in collaboration with your classmates to complete the following exercises - this means sharing ideas and asking each other questions.

Question 1. Let $\boldsymbol{u} = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}$, $\boldsymbol{v} = \begin{bmatrix} -6 \\ 1 \\ 7 \end{bmatrix}$, $\boldsymbol{w} = \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix}$, and $\boldsymbol{z} = \begin{bmatrix} 3 \\ 7 \\ -5 \end{bmatrix}$.

(a) Are the sets $\{\boldsymbol{u}, \boldsymbol{v}\}$, $\{\boldsymbol{u}, \boldsymbol{w}\}$, $\{\boldsymbol{u}, \boldsymbol{z}\}$, and $\{\boldsymbol{w}, \boldsymbol{z}\}$ each linearly independent? Why or why not?

(b) Is the set $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}, \boldsymbol{z}\}$ linearly independent? Why or why not?

Question 2. Suppose you have a partner who is given the matrix of a linear transformation $\mathbb{R}^5 \to \mathbb{R}^6$. They are not allowed to tell you what the matrix is, but they can answer questions about what it does. In order to reconstitute the matrix, how many questions do you need answered? What are they?

Question 3. Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that corresponds to counterclockwise rotation around the origin by an angle θ. What is its matrix?

Question 4. Suppose a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ satisfies all of the following:

$$T\begin{bmatrix}1\\0\\1\\0\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix}, \qquad T\begin{bmatrix}0\\1\\1\\0\end{bmatrix} = \begin{bmatrix}0\\2\\3\end{bmatrix}, \qquad T\begin{bmatrix}0\\1\\1\\1\end{bmatrix} = \begin{bmatrix}2\\1\\2\end{bmatrix}, \qquad T\begin{bmatrix}1\\0\\0\\1\end{bmatrix} = \begin{bmatrix}3\\-1\\-1\end{bmatrix}$$

What is its matrix?

Question 5. Consider the three dimensional shear transformation defined by

$$T_\varepsilon(\boldsymbol{u}) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & \varepsilon \end{bmatrix} \boldsymbol{u}, \qquad \boldsymbol{u} \in \mathbb{R}^3$$

where ε is a parameter satisfying $0 \leq \varepsilon \leq 1$.

(a) Describe the domain and range of the function $\boldsymbol{u} \to T_\varepsilon(\boldsymbol{u})$. How does the range of T_ε depend on the value of the parameter ε?

(b) Let $\boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, assume that $\varepsilon \neq 0$, and solve the equation $T_\varepsilon(\boldsymbol{u}) = \boldsymbol{b}$ using back substitution. What happens to the solution when ε becomes very small, i.e. when $\varepsilon \to 0$?

(c) Suppose that one of the entries in the matrix of $T_\varepsilon(\boldsymbol{u})$ is perturbed, so that

$$\widehat{T}_\varepsilon(\boldsymbol{u}) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1+10^{-10} \\ 0 & 0 & \varepsilon \end{bmatrix}$$

Solve the equation $\widehat{T}_\varepsilon(\widehat{\boldsymbol{u}}) = \boldsymbol{b}$ and compare the computed solution $\widehat{\boldsymbol{u}}$ with the solution $\boldsymbol{u}$ from part (b). Is $\widehat{\boldsymbol{u}}$ an accurate approximation to $\boldsymbol{u}$?

(d) Suppose that the right hand side $\boldsymbol{b}$ is perturbed instead, so that

$$\overline{\boldsymbol{b}} = \begin{bmatrix} 1 \\ 2 \\ 1+10^{-10} \end{bmatrix}$$

Solve the equation $T_\varepsilon(\overline{\boldsymbol{u}}) = \overline{\boldsymbol{b}}$ and compare the computed solution $\overline{\boldsymbol{u}}$ with the true solution $\boldsymbol{u}$.

When the parameter ε is very small, the shear transformation $T_\varepsilon(\boldsymbol{u})$ is an example of an ill-conditioned linear transformation. Ill-conditioned linear systems are difficult to solve accurately on the computer because small round-off errors committed during storage and manipulation of the coefficients and right-hand side can lead to large errors in the computed solution.

Question 6. Use your answer from Question 3 and composition of linear transformations to show that:

$$\cos(\theta_1 + \theta_2) = \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)$$
$$\sin(\theta_1 + \theta_2) = \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2)$$

Answer to Question 1.

(a) For sets containing only two vectors, you can check for linear dependence / independence by just checking whether one of the vectors is a multiple of the other.

For each of the pairs:

$$\{\boldsymbol{u},\boldsymbol{v}\}, \qquad \{\boldsymbol{u},\boldsymbol{w}\}, \qquad \{\boldsymbol{u},\boldsymbol{z}\}, \qquad \{\boldsymbol{w},\boldsymbol{z}\}$$

$$\left\{\begin{bmatrix}3\\2\\-4\end{bmatrix},\begin{bmatrix}-6\\1\\7\end{bmatrix}\right\}, \quad \left\{\begin{bmatrix}3\\2\\-4\end{bmatrix},\begin{bmatrix}0\\-5\\2\end{bmatrix}\right\}, \quad \left\{\begin{bmatrix}3\\2\\-4\end{bmatrix},\begin{bmatrix}3\\7\\-5\end{bmatrix}\right\}, \quad \left\{\begin{bmatrix}0\\-5\\2\end{bmatrix},\begin{bmatrix}3\\7\\-5\end{bmatrix}\right\}$$

We can see that this is not case, so

$$\boxed{\text{All four of } \{\boldsymbol{u},\boldsymbol{v}\},\ \{\boldsymbol{u},\boldsymbol{w}\},\ \{\boldsymbol{u},\boldsymbol{z}\}, \text{ and } \{\boldsymbol{w},\boldsymbol{z}\} \text{ are linearly indepedent sets.}}$$

(b) For this problem, since we have more than three vectors in a three-dimensional space, the set has to be linearly dependent. In case you don't understand why this is true, here's an explanation:

The set $\{\boldsymbol{u},\boldsymbol{v},\boldsymbol{w},\boldsymbol{z}\}$ is linearly dependent if there is a non-trivial solution to the system of equations:

$$c_1\boldsymbol{u} + c_2\boldsymbol{v} + c_3\boldsymbol{w} + c_4\boldsymbol{z} = 0 \tag{1}$$

The augmented matrix for this system is:

$$\left[\begin{array}{rrrr|r} 3 & -6 & 0 & 3 & 0\\ 2 & 1 & -5 & 7 & 0\\ -4 & 7 & 2 & -5 & 0 \end{array}\right]$$

Because the rightmost column is all zeros, the system will always be consistent (i.e. have at least one solution.)

Since there are only three rows, when we reduce this matrix, there will be at most three pivots. With four columns, that means there has to be at least one column that is not a pivot column, which would correspond to a free variable. Since there is a free variable and the system is consistent, we can just chose the value of our free variable to be something other than zero and get a non-zero solution to (1). Therefore,

$$\boxed{\text{The set } \{\boldsymbol{u},\boldsymbol{v},\boldsymbol{w},\boldsymbol{z}\} \text{ is linearly dependent.}}$$

Answer to Question 2. For the sake of convenience, let's call this linear transformation L.
To reconstitute the matrix, you need to know what each of the columns are. To figure out the first column, you can just ask for the result of $L(\boldsymbol{e}_1)$. For the second column, you can ask for the result of $L(\boldsymbol{e}_2)$, and so on.
Then the matrix is just

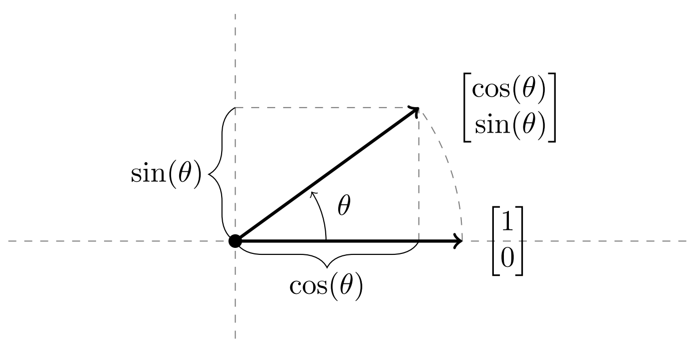
$$\begin{bmatrix} L(\boldsymbol{e}_1) & L(\boldsymbol{e}_2) & L(\boldsymbol{e}_3) & L(\boldsymbol{e}_4) & L(\boldsymbol{e}_5) \end{bmatrix}$$

All together, you have to ask five questions:

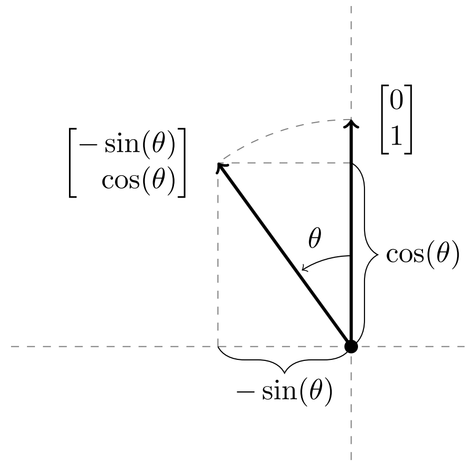
$$\boxed{\text{What are } L(\boldsymbol{e}_1),\ L(\boldsymbol{e}_2),\ L(\boldsymbol{e}_3),\ L(\boldsymbol{e}_4),\ \text{and } L(\boldsymbol{e}_5)?}$$

Note: You don't have to use $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$, $\boldsymbol{e}_4$, and $\boldsymbol{e}_5$. You can actually use any set of 5 vectors that span $\mathbb{R}^5$ and still figure out the resulting matrix, it will just take more work. See Question 4 for an example of how to do this in $\mathbb{R}^4$.

Answer to Question 3. To figure out what this matrix is, we need to ask two questions: What happens when we apply this transformation to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and what happens when we apply this transformation to $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.



The first vector $\boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ corresponds to a horizontal vector with length one, *i.e.* a vector starting from the origin whose endpoint is $x = 1$, $y = 0$. When we rotate counterclockwise around the origin by an angle θ, the endpoint is now at $x = \cos(\theta)$, $y = \sin(\theta)$ (see picture above). So the first column of the matrix is $\begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$.



The second vector $\boldsymbol{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ corresponds to a vertical vector with length one, *i.e.* a vector starting from the origin whose endpoint is $x = 0$, $y = 1$. When we rotate counterclockwise around the origin by an angle θ, the endpoint is now at $x = -\sin(\theta)$, $y = \cos(\theta)$ (see picture above). So the second column of the matrix is $\begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}$.

Putting this all together, the matrix is:

$$\boxed{\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}}$$

Answer to Question 4. To figure out the matrix of T, we will need to figure out its columns, which are:

$$T\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad T\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad T\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad T\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

We can accomplish this using the given values, and the rules for linear transformations. For instance, because T is linear:

$$T\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = T\left(\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}\right) = T\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - T\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

and both of those terms on the right hand side are given to us by the problem, so

$$T\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$$

which gives us the fourth column of the matrix of T.

We can calculate the rest of the columns in a similar way:

$$T\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = T\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - T\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$T\begin{bmatrix}0\\0\\1\\0\end{bmatrix} = T\begin{bmatrix}1\\0\\1\\0\end{bmatrix} - T\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix} - \begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\1\\1\end{bmatrix}$$

$$T\begin{bmatrix}0\\1\\0\\0\end{bmatrix} = T\begin{bmatrix}0\\1\\1\\0\end{bmatrix} - T\begin{bmatrix}0\\0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\2\\3\end{bmatrix} - \begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}0\\1\\2\end{bmatrix}$$

Putting this all together, the matrix is:

$$\boxed{T = \begin{bmatrix}1 & 0 & 0 & 2\\0 & 1 & 1 & -1\\0 & 2 & 1 & -1\end{bmatrix}}$$

Answer to Question 5.

(a) Since T_ε is a 3×3 matrix, this function takes in vectors from $\mathbb{R}^3$, and outputs vectors in $\mathbb{R}^3$. So the domain is all of $\mathbb{R}^3$. The range will depend upon the value of the parameter ε.

If $\varepsilon > 0$, then we can see that the system of equations:

$$T_\varepsilon(\boldsymbol{u}) = \begin{bmatrix}1 & 0 & 1\\0 & 1 & 1\\0 & 0 & \varepsilon\end{bmatrix}\boldsymbol{u} = \boldsymbol{b}$$

has a solution $\boldsymbol{u}$ for every possible $\boldsymbol{b} \in \mathbb{R}^3$. This means that the range is all of $\mathbb{R}^3$.

If $\varepsilon = 0$, then the range is the span of the columns:

$$\text{Span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}\right\}$$

we can easily see that the third vector is redundant, so we can also write this as:

$$\text{Span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$$

which is the set of all points in $\mathbb{R}^3$ that lie on the xy-plane.

Putting this all together,

$$\boxed{\text{Domain } = \mathbb{R}^3, \qquad \text{Range } = \begin{cases}\mathbb{R}^3, & \varepsilon > 0\\ \text{Span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}, & \varepsilon = 0\end{cases}}$$

(b) For this problem, our augmented matrix is:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & \varepsilon & 1 \end{array}\right]$$

We can solve this using back substitution. First, we solve the third equation for x_3:

$$\begin{aligned} \varepsilon x_3 &= 1 \\ x_3 &= \frac{1}{\varepsilon} \end{aligned}$$

Then we plug this into our second equation and solve for x_2:

$$\begin{aligned} x_2 + x_3 &= 2 \\ x_2 &= 2 - \frac{1}{\varepsilon} \end{aligned}$$

And plug into the first equation and solve for x_1:

$$\begin{aligned} x_1 + x_3 &= 1 \\ x_1 &= 1 - \frac{1}{\varepsilon} \end{aligned}$$

All together, our solution is

$$\boxed{\boldsymbol{u} = \begin{bmatrix} 1 - 1/\varepsilon \\ 2 - 1/\varepsilon \\ 1/\varepsilon \end{bmatrix}}$$

In the limit as $\varepsilon \to 0$, $\frac{1}{\varepsilon} \to +\infty$, so

$$\boxed{\lim_{\varepsilon \to 0} \boldsymbol{u} = \begin{bmatrix} -\infty \\ -\infty \\ +\infty \end{bmatrix}}$$

(c) For this problem, our augmented matrix is:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 + 10^{-10} & 2 \\ 0 & 0 & \varepsilon & 1 \end{array}\right]$$

We can solve this using back substitution. First, we solve the third equation for $\hat{x}_3$:

$$\begin{aligned} \varepsilon \hat{x}_3 &= 1 \\ \hat{x}_3 &= \frac{1}{\varepsilon} \end{aligned}$$

Then we plug this into our second equation and solve for $\hat{x}_2$:

$$\begin{aligned} \hat{x}_2 + \left(1 + 10^{-10}\right) \hat{x}_3 &= 2 \\ \hat{x}_2 &= 2 - \frac{1 + 10^{-10}}{\varepsilon} \end{aligned}$$

And plug into the first equation and solve for $\hat{x}_1$:

$$\hat{x}_1 + \hat{x}_3 = 1$$
$$\hat{x}_1 = 1 - \frac{1}{\varepsilon}$$

All together, our solution is

$$\boxed{\widehat{\boldsymbol{u}} = \begin{bmatrix} 1 - 1/\varepsilon \\ 2 - \frac{1+10^{-10}}{\varepsilon} \\ 1/\varepsilon \end{bmatrix}}$$

To check if this is a good approximation, we can compute the difference

$$\boldsymbol{u} - \widehat{\boldsymbol{u}} = \begin{bmatrix} 1 - 1/\varepsilon \\ 2 - 1/\varepsilon \\ 1/\varepsilon \end{bmatrix} - \begin{bmatrix} 1 - 1/\varepsilon \\ 2 - \frac{1+10^{-10}}{\varepsilon} \\ 1/\varepsilon \end{bmatrix} = \begin{bmatrix} 0 \\ 10^{-10}/\varepsilon \\ 0 \end{bmatrix}$$

So we see that $\widehat{\boldsymbol{u}}$ is a good approximation for $\boldsymbol{u}$ for large values of ε, but as $\varepsilon \to 0$, the absolute error will become very large. (The relative error, however, is still small.)

(d) For this problem, our augmented matrix is:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & \varepsilon & 1 + 10^{-10} \end{array}\right]$$

We can solve this using back substitution. First, we solve the third equation for $\overline{x}_3$:

$$\varepsilon \overline{x}_3 = 1 + 10^{-10}$$
$$\overline{x}_3 = \frac{1 + 10^{-10}}{\varepsilon}$$

Then we plug this into our second equation and solve for $\overline{x}_2$:

$$\overline{x}_2 + \overline{x}_3 = 2$$
$$\overline{x}_2 = 2 - \frac{1 + 10^{-10}}{\varepsilon}$$

And plug into the first equation and solve for $\overline{x}_1$:

$$\overline{x}_1 + \overline{x}_3 = 1$$
$$\overline{x}_1 = 1 - \frac{1 + 10^{-10}}{\varepsilon}$$

All together, our solution is

$$\boxed{\overline{\boldsymbol{u}} = \begin{bmatrix} 1 - \frac{1+10^{-10}}{\varepsilon} \\ 2 - \frac{1+10^{-10}}{\varepsilon} \\ \frac{1+10^{-10}}{\varepsilon} \end{bmatrix}}$$

To check if this is a good approximation, we can compute the difference

$$\boldsymbol{u} - \overline{\boldsymbol{u}} = \begin{bmatrix} 1 - 1/\varepsilon \\ 2 - 1/\varepsilon \\ 1/\varepsilon \end{bmatrix} - \begin{bmatrix} 1 - \frac{1+10^{-10}}{\varepsilon} \\ 2 - \frac{1+10^{-10}}{\varepsilon} \\ \frac{1+10^{-10}}{\varepsilon} \end{bmatrix} = \begin{bmatrix} 10^{-10}/\varepsilon \\ 10^{-10}/\varepsilon \\ 10^{-10}/\varepsilon \end{bmatrix}$$

So again we see that $\overline{\boldsymbol{u}}$ is a good approximation for $\boldsymbol{u}$ for large values of ε, but as $\varepsilon \to 0$, the absolute error will become very large.

Answer to Question 6. First, let's define R_θ to be the linear transformation corresponding to counterclockwise rotation by an angle θ.

The key idea here is that rotating all at once by an angle $\theta_1 + \theta_2$ gives exactly the same as rotating first by an angle θ_1, then by an angle θ_2. In mathematical notation, this would mean that for any vector $\boldsymbol{v} \in \mathbb{R}^2$,

$$R_{\theta_1+\theta_2}(\boldsymbol{v}) = R_{\theta_2}\left(R_{\theta_1}(\boldsymbol{v})\right)$$

which we could also phrase as saying that $R_{\theta_1+\theta_2}$ is the same as the composition of R_{θ_1} with R_{θ_2}:

$$R_{\theta_1+\theta_2} = R_{\theta_2} \circ R_{\theta_1}$$

Since these two linear transformations are the same, they should have the same matrix. To figure out what this matrix is, we can first try to calculate the first column by plugging in $\boldsymbol{e}_1$ into these transformations. For the first transformation, we just plug in $\theta_1 + \theta_2$ in place of our answer from Question 3 to get

$$R_{\theta_1+\theta_2}(\boldsymbol{e}_1) = \begin{bmatrix} \cos(\theta_1 + \theta_2) \\ \sin(\theta_1 + \theta_2) \end{bmatrix}$$

For the other transformation, we use our answer from Question 3 to get

$$R_{\theta_2}\left(R_{\theta_1}(\boldsymbol{e}_1)\right) = R_{\theta_2}\left(\begin{bmatrix} \cos(\theta_1) \\ \sin(\theta_1) \end{bmatrix}\right)$$

So now we just multiply $\begin{bmatrix} \cos(\theta_1) \\ \sin(\theta_1) \end{bmatrix}$ with the matrix for rotation by θ_2,

$$\begin{aligned} R_{\theta_2}\left(\begin{bmatrix} \cos(\theta_1) \\ \sin(\theta_1) \end{bmatrix}\right) &= \begin{bmatrix} \cos(\theta_2) & -\sin(\theta_2) \\ \sin(\theta_2) & \cos(\theta_2) \end{bmatrix} \cdot \begin{bmatrix} \cos(\theta_1) \\ \sin(\theta_1) \end{bmatrix} \\ &= \begin{bmatrix} \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) \\ \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2) \end{bmatrix} \end{aligned}$$

So comparing our two different answers for the first column,

$$R_{\theta_1+\theta_2}(\boldsymbol{e}_1) = R_{\theta_2}\left(R_{\theta_1}(\boldsymbol{e}_1)\right)$$

$$\boxed{\begin{bmatrix} \cos(\theta_1 + \theta_2) \\ \sin(\theta_1 + \theta_2) \end{bmatrix} = \begin{bmatrix} \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) \\ \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2) \end{bmatrix}}$$

which gives us the two desired formulas.