Math 2940 Worksheet

Solutions to Linear Systems

Week 3

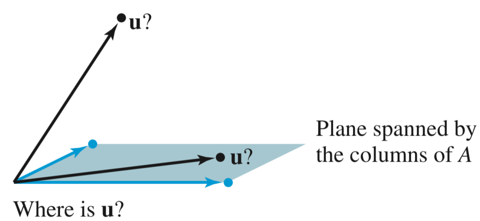
September 12th, 2019

This worksheet covers material from **Section 1.3-1.5**. Please work in collaboration with your classmates to complete the following exercises - this means sharing ideas and asking each other questions.

Question 1. Let $\boldsymbol{u} = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}$ and $A = \begin{bmatrix} 3 & -5 \\ -2 & 6 \\ 1 & 1 \end{bmatrix}$.

Is $\boldsymbol{u}$ in the plane spanned by the columns of A?

Why or why not?



Question 2. Construct a 3×3 matrix A and vectors $\boldsymbol{b}$ and $\boldsymbol{c}$ in $\mathbb{R}^3$ so that $A\boldsymbol{x} = \boldsymbol{b}$ has a solution but $A\boldsymbol{x} = \boldsymbol{c}$ does not.

Question 3. Suppose A is an $m \times n$ matrix. In which of the following cases does $A\boldsymbol{x} = \boldsymbol{b}$ have a solution for every $\boldsymbol{b}$ in $\mathbb{R}^m$?

(a) Every $\boldsymbol{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.

(b) The columns of A span $\mathbb{R}^m$.

(c) A has a pivot position in each row.

(d) The augmented matrix $[A\ \boldsymbol{b}]$ has a pivot position in each row.

(e) A has a pivot position in each column.

(f) The augmented matrix $[A\ \boldsymbol{b}]$ has a pivot position in each column.

Question 4. Each of the following equations determines a plane in $\mathbb{R}^3$. Do the two planes intersect? If so, describe their intersection.

$$\begin{aligned} x_1 + 4x_2 - 5x_3 &= 0 \\ 2x_1 - x_2 + 8x_3 &= 9 \end{aligned}$$

Question 5. Write a system of linear equations and the reduced echelon form of the corresponding augmented matrix that meets the requirements described in the table. If no such system exists, state this and explain why.

	No intersection	Intersects in a point	Intersects in a line	Intersects in a plane
2 equations & 2 unknowns				
2 equations & 3 unknowns				
3 equations & 2 unknowns				
3 equations & 3 unknowns				

(b) Can you make any generalizations from these examples and the strategies you used to create them?

Answer to Question 1. We know that $\boldsymbol{u}$ is in the span of the columns of A, if there exist constants c_1 and c_2 such that:

$$\boldsymbol{u} = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -5 \\ 6 \\ 1 \end{bmatrix}$$

which is equivalent to asking if the linear system of equations

$$\begin{aligned} 3c_1 - 5c_2 &= 2 \\ -2c_1 + 6c_2 &= -3 \\ c_1 + c_2 &= 2 \end{aligned}$$

is consistent or not.

The corresponding augmented matrix is:

$$\left[\begin{array}{cc|c} 3 & -5 & 2 \\ -2 & 6 & -3 \\ 1 & 1 & 2 \end{array}\right]$$

To see whether or not the system is consistent, we need to row reduce it to echelon form. First we can swap the first and third rows,

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ -2 & 6 & -3 \\ 3 & -5 & 2 \end{array}\right] \begin{array}{l} \leftarrow \text{Row } 3 \\ \\ \leftarrow \text{Row } 1 \end{array}$$

then

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 8 & 1 \\ 0 & -8 & -4 \end{array}\right] \begin{array}{l} \\ \leftarrow \text{Row } 2 + 2 \cdot (\text{ Row } 1) \\ \leftarrow \text{Row } 3 - 3 \cdot (\text{ Row } 1) \end{array}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 8 & 1 \\ 0 & 0 & -3 \end{array}\right] \begin{array}{l} \\ \\ \leftarrow \text{Row } 3 + 2 \cdot (\text{ Row } 2) \end{array}$$

From here, we can already see that the augmented matrix is inconsistent, so there is no need to reduce any further. Because the system is inconsistent,

$\boxed{\boldsymbol{u} \text{ is not in the plane spanned by the columns of } A.}$

Answer to Question 2. To make this simpler, we can just consider matrices that are in reduced echelon form, since those allow us to easily see that the solutions are.

To create a situation where $A\boldsymbol{x} = \boldsymbol{c}$ does *not* have a solution, we will need to choose a coefficient matrix A that does not have a pivot in every row, *e.g.*:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Then for our vectors, we can choose

$$\boldsymbol{b} = \begin{bmatrix}1\\1\\0\end{bmatrix}, \qquad \text{and} \qquad \boldsymbol{c} = \begin{bmatrix}1\\1\\1\end{bmatrix}$$

Clearly we can see from the augmented matrices that

$$[A \mid \boldsymbol{b}] = \left[\begin{array}{ccc|c}1 & 0 & 1 & 1\\0 & 1 & 1 & 1\\0 & 0 & 0 & 0\end{array}\right] \qquad \text{has at least one solution}$$

and

$$[A \mid \boldsymbol{c}] = \left[\begin{array}{ccc|c}1 & 0 & 1 & 1\\0 & 1 & 1 & 1\\0 & 0 & 0 & 1\end{array}\right] \qquad \text{does not have any solutions}$$

So one possible answer is:

$$\boxed{A = \begin{bmatrix}1 & 0 & 1\\0 & 1 & 1\\0 & 0 & 0\end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix}1\\1\\0\end{bmatrix}, \qquad \text{and} \qquad \boldsymbol{c} = \begin{bmatrix}1\\1\\1\end{bmatrix}}$$

Answer to Question 3.

(a) $\boxed{\text{True.}}$ Let's write the columns of A as vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, ...\boldsymbol{a}_n$. If $\boldsymbol{b}$ is a linear combination of the columns of A, that means there exist constants $c_1, c_2, ...c_n$, such that

$$c_1\boldsymbol{a}_1 + c_2\boldsymbol{a}_2 + ...c_n\boldsymbol{a}_n = \boldsymbol{b}$$

which is equivalent to

$$\begin{bmatrix}\boldsymbol{a}_1 & \boldsymbol{a}_2 & ... & \boldsymbol{a}_n\end{bmatrix} \cdot \begin{bmatrix}c_1\\c_2\\\vdots\\c_n\end{bmatrix} = A\begin{bmatrix}c_1\\c_2\\\vdots\\c_n\end{bmatrix} = \boldsymbol{b}$$

Which means that $\boldsymbol{x} = \begin{bmatrix}c_1\\c_2\\\vdots\\c_n\end{bmatrix}$ is a solution to $A\boldsymbol{x} = \boldsymbol{b}$

(b) $\boxed{\text{True.}}$ Since the columns of A span $\mathbb{R}^m$, this means that every $\boldsymbol{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A, which is exactly the same as part (a).

(c) $\boxed{\text{True.}}$ The only way that $A\boldsymbol{x} = \boldsymbol{b}$ would not have a solution (*i.e.* be inconsistent) is if when we get to reduced echelon form, we get a row that looks like:

$$\left[\begin{array}{cccc|c}0 & 0 & ... & 0 & 1\end{array}\right]$$

But if there is a pivot position in each row of A, then we cannot possibly have a row of all zero's on the left like we wrote above. If it's impossible for our system to be inconsistent, then it must be consistent, *i.e* $A\boldsymbol{x} = \boldsymbol{b}$ has at least one solution for every possible $\boldsymbol{b}$

(d) $\boxed{\text{False.}}$ In this case, it is possible for A to have a pivot position in the final column, in which case the system would be inconsistent. As a counter-example, consider a case where the augmented matrix is:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

This augmented matrix has a pivot in each row (the 1's), but the last row makes the system inconsistent.

(e) $\boxed{\text{False.}}$ If A was a square matrix, then part (e) would be the same as part (c), and the answer would be True. However, the question says that A is an $m \times n$ matrix, so we could have a tall matrix that corresponds to an overdetermined system. As a counter example, consider the case:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

where A has a pivot in both columns, but the system clearly has no solution, since the third row corresponds to $0 = 1$.

(f) $\boxed{\text{False.}}$ We can actually re-use the same counterexample as in part (e), since the augmented matrix

$$[A \mid b] = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

clearly corresponds to an inconsistent system.

Answer to Question 4. To find the intersection, we just consider this as a linear system, and row reduce the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 4 & -5 & 0 \\ 2 & -1 & 8 & 9 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 4 & -5 & 0 \\ 0 & -9 & 18 & 9 \end{array}\right] \leftarrow \text{Row } 2 \ + (-2) \cdot (\text{ Row } 1)$$

$$\left[\begin{array}{ccc|c} 1 & 4 & -5 & 0 \\ 0 & 1 & -2 & -1 \end{array}\right] \leftarrow (-1/9) \cdot (\text{ Row } 2)$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 4 \\ 0 & 1 & -2 & -1 \end{array}\right] \leftarrow \text{Row } 1 \ + (-4) \cdot (\text{ Row } 2)$$

Here we can see that the system does have solutions, so $\boxed{\text{yes, the two planes intersect}}$. Since there is one free variable, that means the intersection is "one-dimensional," so they actually intersect in a line.

To actually figure out what this line is, we want to write our solution in *parametric vector form*: $\boldsymbol{x} = \boldsymbol{p} + t\boldsymbol{v}$.

To do this, we first use the two equations from our augmented matrix:

$$x_1 + 3x_3 = 4$$
$$x_2 - 2x_3 = -1$$

and write them as:

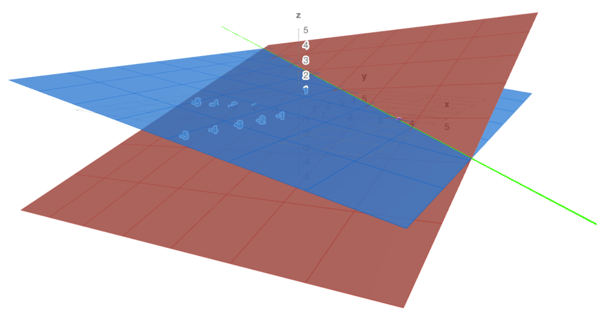
$$x_1 = 4 - 3x_3$$
$$x_2 = -1 + 2x_3$$

Since x_3 is a free variable, we'll also use the equation $x_3 = x_3$ to write the solution set (*i.e.* the intersection) in vector form:

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}}$$

This means the intersection is the line parallel to $\begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$ passing through the point $\begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}$.

In 3D, the planes and intersection looks like this:



where the first plane is in red, the second plane is in blue, and their intersection is in green. You should be able to view an interactive version at: https://www.math3d.org/MA4trebz

Answer to Question 5.
(a) (*Note*: there are many possible examples for most of these, I will just write down the original systems that I came up with, and the corresponding augmented matrix *in reduced echelon form.*)

	No intersection	Intersects in a point	Intersects in a line	Intersects in a plane
2 equations & 2 unknowns	$\begin{aligned} x+y&=0 \\ x+y&=1 \end{aligned}$ $\left[\begin{array}{cc\|c} 1 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$	$\begin{aligned} x+y&=1 \\ y&=1 \end{aligned}$ $\left[\begin{array}{cc\|c} 1 & 0 & 0 \\ 0 & 1 & 1 \end{array}\right]$	$\begin{aligned} x+y&=1 \\ 2x+2y&=2 \end{aligned}$ $\left[\begin{array}{cc\|c} 1 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$	Not possible, too few variables. (Unless you count the silly case where both equations are $0=0$, which I don't)
2 equations & 3 unknowns	$\begin{aligned} x+y+z&=0 \\ x+y+z&=1 \end{aligned}$ $\left[\begin{array}{ccc\|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$	Not possible, if the system is consistent then it has at least one free variable (infinite solutions)	$\begin{aligned} x+y+z&=0 \\ z&=0 \end{aligned}$ $\left[\begin{array}{ccc\|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$	$\begin{aligned} x+y+z&=1 \\ 2x+2y+2z&=2 \end{aligned}$ $\left[\begin{array}{ccc\|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$
3 equations & 2 unknowns	$\begin{aligned} x+y&=0 \\ x+y&=1 \\ 2x+2y&=2 \end{aligned}$ $\left[\begin{array}{cc\|c} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right]$	$\begin{aligned} x+y&=1 \\ y&=1 \\ 2x+2y&=2 \end{aligned}$ $\left[\begin{array}{cc\|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$	$\begin{aligned} x+y&=1 \\ 2x+2y&=2 \\ 3x+3y&=3 \end{aligned}$ $\left[\begin{array}{cc\|c} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$	Not possible, too few variables. (Unless you count the silly case where all three equations are $0=0$, which I don't)
3 equations & 3 unknowns	$\begin{aligned} x+y+z&=0 \\ x+y+z&=1 \\ 2x+2y+2z&=2 \end{aligned}$ $\left[\begin{array}{ccc\|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$	$\begin{aligned} x+y+z&=1 \\ y+z&=1 \\ z&=1 \end{aligned}$ $\left[\begin{array}{ccc\|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$	$\begin{aligned} x+y+z&=1 \\ 2x+2y+2z&=2 \\ z&=0 \end{aligned}$ $\left[\begin{array}{ccc\|c} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$	$\begin{aligned} x+y+z&=1 \\ 2x+2y+2z&=2 \\ 3x+3y+3z&=3 \end{aligned}$ $\left[\begin{array}{ccc\|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

(b) There are many possible generalizations you could draw here, but one useful one is that if an underdetermined system (*i.e.* a system with more unknowns than equations) is consistent, then it must have infinitely many solutions.