# Math 2940 Worksheet

Linear Systems of Equations

Week 2

September 5th, 2019

This worksheet covers material from **Section 1.1-1.3**. Please work in collaboration with your classmates to complete the following exercises - this means sharing ideas and asking each other questions.

**Question 1.** For what values of $h$ and $k$ is the following system consistent?

$$\begin{aligned} 2x_1 - \phantom{3}x_2 &= h \\ -6x_1 + 3x_2 &= k \end{aligned}$$

**Question 2.** Find the general solution of the following system:

$$\begin{aligned} x_1 - 2x_2 - \phantom{5}x_3 + 3x_4 &= 0 \\ -2x_1 + 4x_2 + 5x_3 - 5x_4 &= 3 \\ 3x_1 - 6x_2 - 6x_3 + 8x_4 &= 2 \end{aligned}$$

**Question 3.** Suppose we have the following vectors:

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}, \qquad \boldsymbol{v}_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \qquad \text{and} \qquad \boldsymbol{y} = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}$$

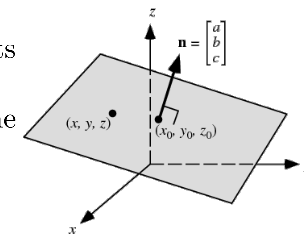
**(a)** Explain what Span $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is in your own words.

**(b)** For what value(s) of $h$ will the vector $y$ be in Span $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$?

**Question 4.** Last week we had the three vectors:

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix}, \qquad \boldsymbol{v}_3 = \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix}$$

and it turned out that these three vectors were all in the same plane. Suppose we wanted to find the actual equation describing this plane.




One way of describing a plane passing through the origin is in terms of its normal vector $\boldsymbol{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. If we have a normal vector, then we can write the equation for the plane as $ax + by + cz = 0$.

**(a)** In order to be normal to $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, and $\boldsymbol{v}_3$, what equations should $a$, $b$, and $c$ satisfy?

**(b)** Solve the system of equations you found in part **(a)** for $a$, $b$, and $c$.

**(c)** Write down an equation for the plane spanned by $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, and $\boldsymbol{v}_3$.

**Question 5.** A company has three separate manufacturing lines, which assemble product A, product B, and product C, respectively. The following chart provides the cost of parts, labor, and overhead required to produce \$100 worth of each product.

| | Parts (\$) | Labor (\$) | Overhead (\$) |
|---|---|---|---|
| Product A | 10 | 50 | 8 |
| Product B | 20 | 40 | 5 |
| Product C | 50 | 30 | 2 |

If the company has budgeted exactly \$150 for parts, \$290 for labor, and \$38 for overhead, then how many \$ worth of each product can they produce? (Hint: write down a vector equation in which the entries within each vector represent the cost of parts, labor, and overhead for a particular product.)

**Answer to Question 1.** We have the linear system

$$\begin{aligned} 2x_1 - \phantom{3}x_2 &= h \\ -6x_1 + 3x_2 &= k \end{aligned}$$

The augmented matrix for this system is:

$$\left[\begin{array}{rr|r} 2 & -1 & h \\ -6 & 3 & k \end{array}\right]$$

To see whether or not this system is consistent, we want to row reduce this matrix to echelon form. To do this, we add 3 times the first row to the second row as follows:

$$\left[\begin{array}{rr|r} 2 & -1 & h \\ 0 & 0 & k+3h \end{array}\right] \begin{array}{l} \\ \leftarrow \text{ Row } 2 + (3) \cdot \text{Row } 1 \end{array}$$

The last row is now equivalent to the equation $0 = k + 3h$, so the system is consistent if and only if this equation is satisfied.

$\boxed{\text{The system is consistent whenever } k + 3h = 0}$

**Answer to Question 2.** The augmented matrix for this system is

$$\left[\begin{array}{rrrr|r} 1 & -2 & -1 & 3 & 0 \\ -2 & 4 & 5 & -5 & 3 \\ 3 & -6 & -6 & 8 & 2 \end{array}\right]$$

To find the general solution, we now try to row reduce the matrix to echelon form

$$\left[\begin{array}{rrrr|r} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 3 & -6 & -6 & 8 & 2 \end{array}\right] \begin{array}{l} \\ \leftarrow \text{ Row } 2 + (2) \cdot \text{Row } 1 \\ \\ \end{array}$$

$$\left[\begin{array}{rrrr|r} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & -3 & -1 & 2 \end{array}\right] \begin{array}{l} \\ \\ \leftarrow \text{ Row } 3 + (-3) \cdot \text{Row } 1 \end{array}$$

$$\left[\begin{array}{rrrr|r} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & 0 & 0 & 5 \end{array}\right] \begin{array}{l} \\ \\ \leftarrow \text{ Row } 3 + \text{Row } 2 \end{array}$$

Now, we can see that the last row is the equation $0 = 5$, which is clearly inconsistent, so

$\boxed{\text{The system has no solutions}}$

**Answer to Question 3.**
**(a)** Span$\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ refers to the set of all possible linear combinations of $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, and $\boldsymbol{v}_3$. In other words, the set of all possible vectors $\boldsymbol{v}$ that we can write as $\boldsymbol{v} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + c_3\boldsymbol{v}_3$.

If you think of these three vectors as being our "modes of transportation" from last week's worksheet, than this would be all the locations we can reach by using those modes of transportation.

**(b)** The vector $\boldsymbol{y}$ is in Span$\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ if there exist $c_1$, $c_2$, and $c_3$ such that:

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + c_3\boldsymbol{v}_3 = \boldsymbol{y}$$

Plugging in the values for these vectors,

$$c_1 \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix} + c_3 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}$$

This vector equation is equivalent to the following linear system of equations:

$$\begin{aligned} c_1 + 5c_2 - 3c_3 &= -4 \\ -c_1 - 4c_2 + \phantom{3}c_3 &= 3 \\ -2c_1 - 7c_2 \phantom{{}+3c_3} &= h \end{aligned}$$

whose augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & -7 & 0 & h \end{array}\right]$$

which we can reduce as follows:

$$\left[\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ -2 & -7 & 0 & h \end{array}\right] \leftarrow \text{Row } 2 + \text{Row } 1$$

$$\left[\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 3 & -6 & h-8 \end{array}\right] \leftarrow \text{Row } 3 + (2) \cdot \text{Row } 1$$

$$\left[\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & h-5 \end{array}\right] \leftarrow \text{Row } 3 + (-3) \cdot \text{Row } 2$$

Here we can see that this system is consistent if and only if $h = 5$. So

$\boxed{\boldsymbol{y} \text{ is in Span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\} \text{ for } h = 5 \text{ only.}}$

**Answer to Question 4.**

**(a)** For $\boldsymbol{n}$ to be a normal vector for the plane containing $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, and $\boldsymbol{v}_3$, it needs to be perpendicular to all three of those. We can enforce this by making sure their dot products are all equal to zero:

$$\boldsymbol{n} \cdot \boldsymbol{v}_1 = 0, \qquad \boldsymbol{n} \cdot \boldsymbol{v}_2 = 0, \qquad \boldsymbol{n} \cdot \boldsymbol{v}_3 = 0$$

Using our values for $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, and $\boldsymbol{v}_3$, and letting $\boldsymbol{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$,

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0, \qquad \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix} = 0, \qquad \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix} = 0$$

Which gives us a linear system of three equations:

$$\boxed{\begin{aligned} a + b + c &= 0 \\ 6a + 3b + 8c &= 0 \\ 4a + b + 6c &= 0 \end{aligned}}$$

**(b)** Using our system from **(a)** , the augmented matrix is:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 6 & 3 & 8 & 0 \\ 4 & 1 & 6 & 0 \end{array}\right]$$

Which we can row reduce as follows:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -3 & 2 & 0 \\ 4 & 1 & 6 & 0 \end{array}\right] \begin{array}{l} \\ \leftarrow \text{Row } 2 + (-6) \cdot \text{Row } 1 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -3 & 2 & 0 \\ 0 & -3 & 2 & 0 \end{array}\right] \begin{array}{l} \\ \\ \leftarrow \text{Row } 3 + (-4) \cdot \text{Row } 1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -3 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \\ \leftarrow \text{Row } 3 + (-1) \cdot \text{Row } 2 \end{array}$$

From this we can see that there are infinitely many solutions $(a, b, c)$, we just need a nonzero one to get a normal vector $\boldsymbol{n}$. We can write this as the system of equations:

$$\begin{aligned} a + b + c &= 0 \\ -3b + 2c &= 0 \end{aligned}$$

Since our pivots are in the first and second columns, we know that $c$ is a free variable, so we can choose any value for $c$ and then solve for $a$ and $b$ to get a solution to the system of equations. To make everything work out cleanly, I will choose $c = 3$, but any other nonzero choice works just fine. Plugging this into the second equation and solving for $b$

$$\begin{aligned} -3b + 6 &= 0 \\ b &= 2 \end{aligned}$$

Plugging $b = 2$ and $c = 3$ into the first equation,

$$\begin{aligned} a + 3 + 2 &= 0 \\ a &= -5 \end{aligned}$$

So one solution (of many) to this system of equations is:

$$\boxed{a = -5, \quad b = 2, \quad c = 3}$$

**(c)** Since the equation for a plane passing through the origin is $ax + by + cz = 0$, we plug in our values for $a$, $b$, and $c$ to get:

$$\boxed{-5x + 2y + 3z = 0}$$

**Answer to Question 5.**

First, let's define our variables. Let $a$ be the amount of product A we produce, $b$ be the amount of product B, and $c$ be the amount of product C, each in units of \$100.

Then, we'll have three equations, one to match the budget for parts:

$$10a + 20b + 50c = 150$$

Another to match the budget for labor:

$$50a + 40b + 30c = 290$$

And one more to match the budget for overhead:

$$8a + 5b + 2c = 38$$

All together, this system of equations is

$$\begin{aligned} 10a + 20b + 50c &= 150 \\ 50a + 40b + 30c &= 290 \\ 8a + \ \ 5b + \ \ 2c &= 38 \end{aligned}$$

So the augmented matrix is

$$\left[\begin{array}{rrr|r} 10 & 20 & 50 & 150 \\ 50 & 40 & 30 & 290 \\ 8 & 5 & 2 & 38 \end{array}\right]$$

Now we want to row reduce this matrix to reduced echelon form. There are many ways of doing this, but I'll show one that keeps the arithmetic relatively manageable. First, let's scale down the first two rows

$$\left[\begin{array}{rrr|r} 1 & 2 & 5 & 15 \\ 5 & 4 & 3 & 29 \\ 8 & 5 & 2 & 38 \end{array}\right] \begin{array}{l} \leftarrow \left(\frac{1}{10}\right) \cdot \text{Row 1} \\ \leftarrow \left(\frac{1}{10}\right) \cdot \text{Row 2} \\ \phantom{x} \end{array}$$

Clearing out the first column,

$$\left[\begin{array}{rrr|r} 1 & 2 & 5 & 15 \\ 0 & -6 & -22 & -46 \\ 8 & 5 & 2 & 38 \end{array}\right] \begin{array}{l} \\ \leftarrow \text{Row } 2 + (-5) \cdot \text{Row } 1 \\ \\ \end{array}$$

$$\left[\begin{array}{rrr|r} 1 & 2 & 5 & 15 \\ 0 & -6 & -22 & -46 \\ 0 & -11 & -38 & -82 \end{array}\right] \begin{array}{l} \\ \\ \leftarrow \text{Row } 3 + (-8) \cdot \text{Row } 1 \end{array}$$

Swapping the second and third rows,

$$\left[\begin{array}{rrr|r} 1 & 2 & 5 & 15 \\ 0 & -11 & -38 & -82 \\ 0 & -6 & -22 & -46 \end{array}\right] \begin{array}{l} \\ \leftarrow \text{Row } 3 \\ \leftarrow \text{Row } 2 \end{array}$$

then

$$\left[\begin{array}{rrr|r} 1 & 2 & 5 & 15 \\ 0 & 1 & 6 & 10 \\ 0 & -6 & -22 & -46 \end{array}\right] \begin{array}{l} \\ \leftarrow \text{Row } 2 + (-2) \cdot \text{Row } 3 \\ \\ \end{array}$$

$$\left[\begin{array}{rrr|r} 1 & 2 & 5 & 15 \\ 0 & 1 & 6 & 10 \\ 0 & 0 & 14 & 14 \end{array}\right] \begin{array}{l} \\ \\ \leftarrow \text{Row } 3 + (6) \cdot \text{Row } 2 \end{array}$$

Now we have the matrix in echelon form, so we can do the backward phase to get it into reduced echelon form as follows:

$$\left[\begin{array}{rrr|r} 1 & 2 & 5 & 15 \\ 0 & 1 & 6 & 10 \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{l} \\ \\ \leftarrow \left(\frac{1}{14}\right) \cdot \text{Row } 3 \end{array}$$

$$\left[\begin{array}{rrr|r} 1 & 2 & 5 & 15 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{l} \\ \leftarrow \text{Row } 2 + (-6) \cdot \text{Row } 3 \\ \\ \end{array}$$

$$\left[\begin{array}{rrr|r} 1 & 2 & 0 & 10 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{l} \leftarrow \text{Row } 1 + (-5) \cdot \text{Row } 3 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{rrr|r} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{l} \leftarrow \text{Row } 1 + (-2) \cdot \text{Row } 2 \\ \\ \\ \end{array}$$

From here we can read off the solution as $a = 2$, $b = 4$, and $c = 1$. Since these are in units of \$100, our final answer is:

$$\boxed{\text{\$200 of Product A,} \quad \text{\$400 of Product B,} \quad \text{\$100 of Product C}}$$