# Math 2940 Worksheet

Complex Eigenvalues, Dynamical Systems

Week 12

November 14th, 2019

This worksheet covers material from **Sections 5.5 - 5.7**. Please work in collaboration with your classmates to complete the following exercises - this means sharing ideas and asking each other questions.

**Question 1.** Show that if $a$ and $b$ are real, then the eigenvalues of $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ are $a \pm bi$, with corresponding eigenvectors $\begin{bmatrix} 1 \\ -i \end{bmatrix}$ and $\begin{bmatrix} 1 \\ i \end{bmatrix}$.

**Question 2.** The matrix $A$ below has eigenvalues 1, $\frac{2}{3}$, and $\frac{1}{3}$, with corresponding eigenvectors $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, and $\boldsymbol{v}_3$:

$$A = \frac{1}{9}\begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 5 \end{bmatrix}, \quad \boldsymbol{v}_1 = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \quad \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

(a) Find the solution of the equation $\boldsymbol{x}_{k+1} = A\boldsymbol{x}_k$ if $\boldsymbol{x}_0 = \begin{bmatrix} 1 \\ 11 \\ -2 \end{bmatrix}$.

(b) What happens to the sequence $\{\boldsymbol{x}_k\}$ from part (a) as $k \to \infty$?

(c) Would your answer to part (b) be the same for all values of $\boldsymbol{x}_0$? Why or why not?

**Question 3.** Here we are interested in how the populations of two species change over time. Let $x(t)$ represent the population of prey, and $y(t)$ represent the population of predators.

Suppose that the change in populations over time can be described by the following linear system of differential equations:

$$\begin{aligned} x' &= 7x - y \\ y' &= 3x + 3y \end{aligned}$$

Given the initial conditions $\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, find the solution to this system of differential equations.

**Question 4.** Let $A = \begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix}$.

(a) Construct the general solution of $\boldsymbol{x}' = A\boldsymbol{x}$ involving complex eigenfunctions

(b) Using your answer to part (a), obtain the general real solution

**Question 5.**

(a) Not every matrix is diagonalizable. Consider the matrix $A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$. Compute the eigenvalues of $A$ and compute the corresponding eigenvector. How many distinct eigenvalues and eigenvectors does $A$ have?

(b) On the other hand, consider the matrix $B = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & -1 & 3 \end{bmatrix}$. How many distinct eigenvalues and eigenvectors does $B$ have?

(c) Write down the fundamental solution to the initial value problem

$$\boldsymbol{x}' = A\boldsymbol{x}, \qquad \boldsymbol{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

where $A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$ is the matrix from part (a).

**Answer to Question 1.** To find the eigenvalues, we calculate the roots of the characteristic equation:

$$\begin{aligned}
&\det(A-\lambda I) = \det\left(\begin{bmatrix} a-\lambda & -b \\ b & a-\lambda \end{bmatrix}\right) = 0 \\
&(a-\lambda)^2 + b^2 = 0 \\
&\lambda^2 - 2a\lambda + a^2 + b^2 = 0
\end{aligned}$$

Using the quadratic formula,

$$\lambda = \frac{2a \pm \sqrt{4a^2 - 4(a^2+b^2)}}{2} = \frac{2a \pm \sqrt{-4b^2}}{2}$$

So the eigenvalues are:

$$\boxed{\lambda = a \pm bi}$$

Now, to find the eigenvalue associated with $\lambda_1 = a + bi$, we compute the nullspace of $A - \lambda_1 I$

$$A - (a+bi)I = \begin{bmatrix} a-(a+bi) & -b \\ b & a-(a+bi) \end{bmatrix} = \begin{bmatrix} -bi & -b \\ b & -bi \end{bmatrix}$$

Row reducing,

$$\begin{bmatrix} -bi & -b \\ b & -bi \end{bmatrix} \xrightarrow{i\cdot \text{ R1}} \begin{bmatrix} b & -bi \\ b & -bi \end{bmatrix} \xrightarrow{\text{R2 - R1}} \begin{bmatrix} b & -bi \\ 0 & 0 \end{bmatrix}$$

So the nullspace is described by the equation $bx_1 - bix_2 = 0$. Choosing $x_1 = 1$, it follows that $x_2 = -i$.

$$\boxed{\text{Therefore } \begin{bmatrix} 1 \\ -i \end{bmatrix} \text{ is an eigenvector with eigenvalue } a+bi}$$

Similarly for the other eigenvalue, we compute the nullspace of $A - \lambda_2 I$ as

$$A - (a-bi)I = \begin{bmatrix} a-(a-bi) & -b \\ b & a-(a-bi) \end{bmatrix} = \begin{bmatrix} bi & -b \\ b & bi \end{bmatrix}$$

Row reducing,

$$\begin{bmatrix} bi & -b \\ b & bi \end{bmatrix} \xrightarrow{i\cdot \text{ R1}} \begin{bmatrix} -b & -bi \\ b & bi \end{bmatrix} \xrightarrow{\text{R2 + R1}} \begin{bmatrix} -b & -bi \\ 0 & 0 \end{bmatrix}$$

So the nullspace is described by the equation $-bx_1 - bix_2 = 0$. Choosing $x_1 = 1$, it follows that $x_2 = i$.

$$\boxed{\text{Therefore } \begin{bmatrix} 1 \\ i \end{bmatrix} \text{ is an eigenvector with eigenvalue } a-bi}$$

**Answer to Question 2.**

(a) First we want to write the initial conditions $\boldsymbol{x}_0$ in terms of the eigenvectors, i.e. find constants $c$ such that

$$\boldsymbol{x}_0 = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + c_3\boldsymbol{v}_3$$
$$\begin{bmatrix} 1 \\ 11 \\ -2 \end{bmatrix} = c_1 \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

This is equivalent to the augmented matrix:

$$\left[\begin{array}{ccc|c} -2 & 2 & 1 & 1 \\ 2 & 1 & 2 & 11 \\ 1 & 2 & -2 & -2 \end{array}\right]$$

Row reducing,

$$\left[\begin{array}{ccc|c} -2 & 2 & 1 & 1 \\ 2 & 1 & 2 & 11 \\ 1 & 2 & -2 & -2 \end{array}\right] \xrightarrow{\text{R2 + R1}} \left[\begin{array}{ccc|c} -2 & 2 & 1 & 1 \\ 0 & 3 & 3 & 12 \\ 1 & 2 & -2 & -2 \end{array}\right] \xrightarrow{\text{R3 + }\frac{1}{2}\text{R1}} \left[\begin{array}{ccc|c} -2 & 2 & 1 & 1 \\ 0 & 3 & 3 & 12 \\ 0 & 3 & -3/2 & -3/2 \end{array}\right]$$
$$\xrightarrow{\text{R3-R2}} \left[\begin{array}{ccc|c} -2 & 2 & 1 & 1 \\ 0 & 3 & 3 & 12 \\ 0 & 0 & -9/2 & -27/2 \end{array}\right] \xrightarrow{\frac{-2}{9}\text{R3}} \left[\begin{array}{ccc|c} -2 & 2 & 1 & 1 \\ 0 & 3 & 3 & 12 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow{\frac{1}{3}\text{R2}} \left[\begin{array}{ccc|c} -2 & 2 & 1 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$
$$\xrightarrow{\text{R2-R3}} \left[\begin{array}{ccc|c} -2 & 2 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow{\text{R1-R3}} \left[\begin{array}{ccc|c} -2 & 2 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow{\frac{-1}{2}\text{R1}} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$
$$\xrightarrow{\text{R1+R2}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Therefore the initial condition can be written in terms of the eigenvectors as:

$$\boldsymbol{x}_0 = 2 \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

Using the eigenvalues and eigenvectors, we can compute

$$\boxed{\boldsymbol{x}_k = A^k\boldsymbol{x}_0 = 2(1)^k \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} + \left(\frac{2}{3}\right)^k \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + 3\left(\frac{1}{3}\right)^k \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}}$$

*Note*: This is equivalent to computing the diagonalization $A = PDP^{-1}$, and then finding the solution as $\boldsymbol{x}_k = PD^kP{-}1\boldsymbol{x}_0$.

- Computing $P^{-1}\boldsymbol{x}_0$ is equivalent to finding the coefficients $c_1$, $c_2$, and $c_3$.
- Multiplying by $D^k$ contributes the eigenvalue to the $k$-th power terms above
- Multiplying by $P$ contributes the eigenvectors in the above solution.

(b) As $k \to \infty$, we see that both the $\left(\frac{2}{3}\right)^k$ and $\left(\frac{1}{3}\right)^k$ terms will go to zero, leaving

$$\boxed{\lim_{k\to\infty} \boldsymbol{x}_k = 2\begin{bmatrix}-2\\2\\1\end{bmatrix} = \begin{bmatrix}-4\\4\\2\end{bmatrix}}$$

(c) If we write the initial conditions as $\boldsymbol{x}_0 = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + c_3\boldsymbol{v}_3$, then we can see from the previous problems that

$$\lim_{k\to\infty} \boldsymbol{x}_k = c_1\begin{bmatrix}-2\\2\\1\end{bmatrix}$$

Since $c_1$ clearly does depend upon the value of $\boldsymbol{x}_0$, then

$\boxed{\text{No, our answer would not be the same for all values of } \boldsymbol{x}_0.}$

**Answer to Question 3.** First, we can write this linear system of differential equations in matrix form as

$$\begin{bmatrix}x\\y\end{bmatrix}' = \begin{bmatrix}7 & -1\\3 & 3\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

Now we want to compute the eigenvalues and eigenvectors of $A = \begin{bmatrix}7 & -1\\3 & 3\end{bmatrix}$. Finding the roots of the characteristic equation,

$$\begin{aligned}
\det(A - \lambda I) &= 0\\
\det\left(\begin{bmatrix}7-\lambda & -1\\3 & 3-\lambda\end{bmatrix}\right) &= 0\\
(7-\lambda)(3-\lambda) + 3 &= 0\\
\lambda^2 - 10\lambda + 24 &= 0\\
(\lambda-4)(\lambda-6) &= 0
\end{aligned}$$

So we can see that the two eigenvalues are $\lambda = 4$ and $\lambda = 6$.

To find the eigenvector corresponding to $\lambda = 4$, we find the nullspace of $A - 4I$ by row reducing:

$$A - 4I = \begin{bmatrix}3 & -1\\3 & -1\end{bmatrix} \xrightarrow{\text{R2 - R1}} \begin{bmatrix}3 & -1\\0 & 0\end{bmatrix}$$

Plugging $x_1 = 1$ into $3x_1 - x_2 = 0$, we get that $x_2 = 3$, and therefore

$$\begin{bmatrix}1\\3\end{bmatrix} \text{ is an eigenvector of } A \text{ with eigenvalue } \lambda = 4$$

Similarly, we find the eigenvector corresponding to $\lambda = 6$,

$$A - 6I = \begin{bmatrix}1 & -1\\3 & -3\end{bmatrix} \xrightarrow{\text{R2 - 3}\cdot\text{R1}} \begin{bmatrix}1 & -1\\0 & 0\end{bmatrix}$$

Plugging $x_1 = 1$ into $x_1 - x_2 = 0$, we get that $x_2 = 1$, and therefore

$$\begin{bmatrix}1\\1\end{bmatrix} \text{ is an eigenvector of } A \text{ with eigenvalue } \lambda = 6$$

We can put these together to get that the general solution is:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{6t}$$

for some constants $c_1$ and $c_2$.

Plugging in the initial conditions at $t = 0$, we get that

$$x(0) = c_1 + c_2 = 3$$
$$y(0) = 3c_1 + c_2 = 2$$

Putting this in augmented matrix form and row reducing,

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 3 & 1 & 2 \end{array}\right] \xrightarrow{\text{R2 - R1}} \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & 0 & -1 \end{array}\right] \xrightarrow{\frac{1}{2}\cdot\text{ R2}} \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & 0 & -\frac{1}{2} \end{array}\right] \xrightarrow{\text{R1 - R2}} \left[\begin{array}{cc|c} 0 & 1 & \frac{7}{2} \\ 1 & 0 & -\frac{1}{2} \end{array}\right]$$

From this we can see that $c_1 = -\frac{1}{2}$ and $c_2 = \frac{7}{2}$, so the solution to this initial value problem is

$$\boxed{\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{4t} + \frac{7}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{6t}}$$

**Answer to Question 4.**

(a) First, we find the eigenvalues of $A$ by finding the roots of the characteristic equation:

$$\det(A - \lambda I) = 0$$
$$\det\left(\begin{bmatrix} -3-\lambda & 2 \\ -1 & -1-\lambda \end{bmatrix}\right) = 0$$
$$(-3-\lambda)(-1-\lambda) + 2 = 0$$
$$\lambda^2 + 4\lambda + 5 = 0$$

Using the quadratic formula, the eigenvalues are

$$\lambda = \frac{-4 \pm \sqrt{16-20}}{2} = \frac{-4 \pm \sqrt{-4}}{2} = -2 \pm i$$

First we find the eigenvector associated to $\lambda = -2 + i$

$$A - (-2+i)I = \begin{bmatrix} -1-i & 2 \\ -1 & 1-i \end{bmatrix} \xrightarrow{(-1+i)\cdot\text{ R1}} \begin{bmatrix} 2 & -2+2i \\ -1 & 1-i \end{bmatrix} \xrightarrow{\text{R2 + }\frac{1}{2}\cdot\text{ R1}} \begin{bmatrix} 2 & -2+2i \\ 0 & 0 \end{bmatrix}$$

Plugging $x_2 = 1$ into $2x_1 + (-2+2i)x_2 = 0$, we get that $x_1 = 1 - i$, and therefore

$$\begin{bmatrix} 1-i \\ 1 \end{bmatrix} \text{ is an eigenvector of } A \text{ with eigenvalue } \lambda = -2 + i$$

Now to find the eigenvector associated to $\lambda = -2 - i$

$$A - (-2-i)I = \begin{bmatrix} -1+i & 2 \\ -1 & 1+i \end{bmatrix} \xrightarrow{(-1-i)\cdot\text{ R1}} \begin{bmatrix} 2 & -2-2i \\ -1 & 1+i \end{bmatrix} \xrightarrow{\text{R2 + }\frac{1}{2}\cdot\text{ R1}} \begin{bmatrix} 2 & -2-2i \\ 0 & 0 \end{bmatrix}$$

Plugging $x_2 = 1$ into $2x_1 + (-2-2i)x_2 = 0$, we get that $x_1 = 1+i$, and therefore

$$\begin{bmatrix} 1+i \\ 1 \end{bmatrix} \text{ is an eigenvector of } A \text{ with eigenvalue } \lambda = -2-i$$

Putting these together, the general solution using complex eigenfunctions is:

$$\boxed{\boldsymbol{x}(t) = c_1 \begin{bmatrix} 1-i \\ 1 \end{bmatrix} e^{(-2+i)t} + c_2 \begin{bmatrix} 1+i \\ 1 \end{bmatrix} e^{(-2-i)t}}$$

(b) To obtain the general real solution, we want to split up the first fundamental solution into real and imaginary parts:

$$\begin{bmatrix} 1-i \\ 1 \end{bmatrix} e^{(-2+i)t} = \text{Re}(t) + i \cdot \text{Im}(t)$$

Expanding the left hand side using Euler's formula $e^{i\theta} = \cos(\theta) + i\sin(\theta)$,

$$\begin{aligned} &= \begin{bmatrix} 1-i \\ 1 \end{bmatrix} e^{-2t}\cos(t) + i \begin{bmatrix} 1-i \\ 1 \end{bmatrix} e^{-2t}\sin(t) \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t}\cos(t) + i \begin{bmatrix} -1 \\ 0 \end{bmatrix} e^{-2t}\cos(t) + i \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t}\sin(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-2t}\sin(t) \end{aligned}$$

Rearranging,

$$= \begin{bmatrix} \cos(t) + \sin(t) \\ \cos(t) \end{bmatrix} e^{-2t} + i \begin{bmatrix} \sin(t) - \cos(t) \\ \sin(t) \end{bmatrix} e^{-2t}$$

Therefore the real and imaginary parts are:

$$\text{Re}(t) = \begin{bmatrix} \cos(t) + \sin(t) \\ \cos(t) \end{bmatrix} e^{-2t}, \qquad \text{and} \qquad \text{Im}(t) = \begin{bmatrix} \sin(t) - \cos(t) \\ \sin(t) \end{bmatrix} e^{-2t}$$

And we can use these to construct the general real solution:

$$\boxed{\boldsymbol{x}(t) = c_1 \begin{bmatrix} \cos(t) + \sin(t) \\ \cos(t) \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} \sin(t) - \cos(t) \\ \sin(t) \end{bmatrix} e^{-2t}}$$

**Answer to Question 5.**

(a) First we compute the eigenvalues and eigenvectors of $A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$.

Finding the roots of the characteristic equation,

$$\begin{aligned} \det(A - \lambda I) &= 0 \\ \det\left(\begin{bmatrix} 3-\lambda & -1 \\ 1 & 1-\lambda \end{bmatrix}\right) &= 0 \\ (3-\lambda)(1-\lambda) + 1 &= 0 \\ \lambda^2 - 4\lambda + 4 &= 0 \\ (\lambda - 2)^2 &= 0 \end{aligned}$$

So $\boxed{A \text{ only has the single distinct eigenvalue of } \lambda = 2.}$

Finding the corresponding eigenvector(s),

$$A - 2I = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \xrightarrow{\text{R2 - R1}} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

So the nullspace is described by $x_1 - x_2 = 0$. One possible non-zero solution is $x_1 = x_2 = 1$, so therefore

$$\boxed{\begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ is the only eigenvector of } A.}$$

(b) Now to compute the eigenvalues and eigenvectors of $B = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & -1 & 3 \end{bmatrix}$,

Finding the characteristic equation using a co-factor expansion in the first column,

$$\det(B-\lambda I) = \det\left(\begin{bmatrix} 2-\lambda & -1 & 1 \\ 0 & 2-\lambda & 0 \\ 0 & -1 & 3-\lambda \end{bmatrix}\right) = (2-\lambda)\det\left(\begin{bmatrix} 2-\lambda & 0 \\ -1 & 3-\lambda \end{bmatrix}\right) = (2-\lambda)^2(3-\lambda) = 0$$

So there are $\boxed{\text{two distinct eigenvalues: } \lambda = 2 \text{ and } \lambda = 3.}$

First, we find any eigenvectors associated to $\lambda = 2$,

$$B - 2I = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow{\text{R3 - R1}} \begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So the nullspace is described by the equation $-x_2 + x_3 = 0$.

Because $x_1$ is a free variable, we get a solution of $x_1 = 1$, $x_2 = 0$, and $x_3 = 0$.

Setting $x_1 = 0$ and plugging $x_3 = 1$ into $-x_2 + x_3 = 0$, we get that $x_2 = 1$. Therefore,

$$\boxed{\text{Both } \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \text{ are eigenvectors of } B \text{ with eigenvalue } \lambda = 2}$$

Now, we find any eigenvectors associated to $\lambda = 3$,

$$B - 3I = \begin{bmatrix} -1 & -1 & 1 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \end{bmatrix} \xrightarrow{\text{R3 - R2}} \begin{bmatrix} -1 & -1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{R1 - R2}} \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Setting the free variable $x_3 = 1$, we get that $x_1 = 1$ and $x_2 = 0$, so

$$\boxed{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{ is an eigenvector of } B \text{ with eigenvalue } \lambda = 3}$$

All together, there are three eigenvectors but only two distinct eigenvalues.

(c) Because $A$ only has one eigenvector (see part (a)), we will be unable to fully diagonalize $A$. Thus we will be unable to fully de-couple the equations for $x_1(t)$ and $x_2(t)$.

However, with the right change-of-coordinates, we can still "triangularize" the matrix $A$. This will allow us to solve for $x_1(t)$, and then use our solution for $x_2(t)$ to solve for $x_1(t)$.

First, we will define a change-of-coordinates matrix $P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

(The exact change-of-coordinates is not important, only that one of our columns correspond to the eigenvector we found in part (a).)

We can then easily calculate that $P^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

Defining a new set of variables as $P\boldsymbol{y}(t) = \boldsymbol{x}(t)$, we see that they satisfy the system of equations

$$P\boldsymbol{y}'(t) = AP\boldsymbol{y}(t)$$

Multiplying both sides on the left by $P^{-1}$,

$$\boldsymbol{y}'(t) = P^{-1}AP\boldsymbol{y}(t)$$

Using matrix multiplication, we can compute

$$P^{-1}AP = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$$

Therefore our system of equations is

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

The first equation is simply $y_1' = 2y_1$, which has a solution of

$$y_1(t) = c_1 e^{2t}$$

Plugging that into the second equation, we get

$$y_2' = c_1 e^{2t} + 2y_2$$

This is a first-order linear ODE, so we can solve it using integrating factors. Rearranging,

$$y_2' - 2y_2 = c_1 e^{2t}$$

Multiplying both sides by $e^{-2t}$,

$$e^{-2t} y_2' - 2e^{-2t} y_2 = c_1$$

Using the product rule (in reverse) on the left hand side,

$$\frac{d}{dt}\left(e^{-2t} y_2\right) = c_1$$

Integrating both sides,

$$e^{-2t} y_2 = c_1 t + c_2$$

Multiplying both sides by $e^{2t}$,

$$y_2 = c_1 t e^{2t} + c_2 e^{2t}$$

And therefore our general solution is

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{2t} \\ c_1 t e^{2t} + c_2 e^{2t} \end{bmatrix}$$

Undoing our change-of-coordinates,

$$\boldsymbol{x} = P\boldsymbol{y} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{2t} \\ c_1 t e^{2t} + c_2 e^{2t} \end{bmatrix} = \begin{bmatrix} c_1 t e^{2t} + (c_1 + c_2) e^{2t} \\ c_1 t e^{2t} + c_2 e^{2t} \end{bmatrix}$$

Plugging in the initial conditions $\boldsymbol{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$,

$$\boldsymbol{x}(0) = \begin{bmatrix} c_1(0)e^0 + (c_1 + c_2)e^0 \\ c_1(0)e^0 + c_2 e^0 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

So clearly $c_1 = 1$ and $c_2 = 0$. Therefore the solution is

$$\boxed{\boldsymbol{x} = \begin{bmatrix} te^{2t} + e^{2t} \\ te^{2t} \end{bmatrix}}$$