Math 2940 Worksheet

Eigenvalues and Eigenvectors

Week 11

November 7th, 2019

This worksheet covers material from **Sections 5.1 - 5.3**. Please work in collaboration with your classmates to complete the following exercises - this means sharing ideas and asking each other questions.

Question 1.

(a) Find the characteristic equation and eigenvalues of $A = \begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}$.

(b) Using your answer from part(a), find any eigenvectors of A.

Question 2. Without writing down the matrix, what, if anything, can you say about the eigenvalues and eigenvectors of the following linear transformations?

(a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ stretches images by a factor of 2 horizontally, and by a factor of 3 vertically.

(b) $T : \mathbb{R}^2 \to \mathbb{R}^2$ reflects everything across the x-axis.

(c) $T : \mathbb{R}^2 \to \mathbb{R}^2$ rotates everything around the origin by 90° clockwise.

Question 3. Let $A = \begin{bmatrix} -3 & 12 \\ -2 & 7 \end{bmatrix}$, $\boldsymbol{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, and $\boldsymbol{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Suppose you are told that $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are eigenvectors of A. Use this information to diagonalize A.

Question 4.

(a) Consider the following matrix equation:

$$\begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a_{n-1} \\ a_n \end{bmatrix}$$

Starting with $a_0 = a_1 = 1$, use the above equation to compute some more values of a_n. Do you notice a pattern?

To compute large values of a_n, we can see that:

$$\begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a_{n-1} \\ a_n \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^2 \begin{bmatrix} a_{n-2} \\ a_{n-1} \end{bmatrix} = \cdots = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^n \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

However, for large values of n, it would be very tedious and time-consuming to compute $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^n$ directly.

Instead, we want to diagonalize the matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ as PDP^{-1}.

(b) First, compute the eigenvalues of $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

(c) Show that the eigenvectors of A are $\begin{bmatrix} 2 \\ 1+\sqrt{5} \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1-\sqrt{5} \end{bmatrix}$.

(d) Use your answers from parts (b) and (c) to diagonalize $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^n$.

(e) Combine your answers from the previous parts to show that

$$a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$$

(*Hint*: $A^n = PD^nP^{-1}$)

(f) Given that $\frac{1-\sqrt{5}}{2} \approx -0.618$ and $\frac{1+\sqrt{5}}{2} \approx 1.618$, explain why

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} \approx 1.618$$

(Fun fact: $\varphi = \frac{1+\sqrt{5}}{2}$ is also known as the *golden ratio*.)

Question 5. A quasi upper triangular matrix is a matrix that has 1×1 or 2×2 blocks on the diagonal and 0 below these blocks. Two examples of a 4×4 quasi upper triangular matrix are:

$$\begin{bmatrix} \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & 0 & 0 & \times \end{bmatrix}, \qquad \begin{bmatrix} \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & 0 & \times & \times \\ 0 & 0 & \times & \times \end{bmatrix}.$$

(a) When we solve a linear system with an upper triangular matrix, we can use back substitution to solve it fast. Is it possible to generalize back substitution so that we can also solve linear system with quasi upper triangular matrix fast?

The real Schur decomposition of an $n \times n$ matrix A is to factorize $A = UTU^T$, where $U, T \in \mathbb{R}^{n \times n}$, U is orthogonal, and T is a quasi upper triangular matrix. (Recall that for a square orthogonal matrix V, $V^{-1} = V^T$.)

(b) Are the eigenvalues of A and T related? How can you tell what the eigenvalues of T are?

Comparatively, a complex Schur decomposition of A is $A = UTU^T$, where U is orthogonal, and T is upper triangular.

Answer to Question 1.

(a) First, we find the characteristic equation:

$$\begin{aligned} \det(A - \lambda I) &= 0 \\ \det\left(\begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) &= 0 \\ \det\left(\begin{bmatrix} 10-\lambda & -9 \\ 4 & -2-\lambda \end{bmatrix}\right) &= 0 \\ (10-\lambda)(-2-\lambda) + 36 &= 0 \\ -20 - 8\lambda + \lambda^2 + 36 &= 0 \\ \lambda^2 - 8\lambda + 16 &= 0 \end{aligned}$$

Now to find the eigenvalues, we find the roots of the characteristic equation, which factors as:

$$(\lambda - 4)^2 = 0$$

So $\boxed{\text{the only eigenvalue is } \lambda = 4.}$

(b) To find the eigenvector(s), we want to find a basis for the nullspace of $A - \lambda I$:

$$A - \lambda I = \begin{bmatrix} 10-4 & -9 \\ 4 & -2-4 \end{bmatrix} = \begin{bmatrix} 6 & -9 \\ 4 & -6 \end{bmatrix}$$

Row reducing,

$$\begin{aligned} &\sim \begin{bmatrix} 2 & -3 \\ 4 & -6 \end{bmatrix} \\ &\sim \begin{bmatrix} 2 & -3 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

This means elements of the nullspace of $A - \lambda I$ satisfy $2x_1 - 3x_2 = 0$. So the nullspace can be written in parametric form as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 3/2 \\ 1 \end{bmatrix}$$

i.e. $\text{Null}(A - \lambda I) = \text{Span}\left(\begin{bmatrix} 3/2 \\ 1 \end{bmatrix}\right)$

Therefore, $\boxed{\begin{bmatrix} 3/2 \\ 1 \end{bmatrix} \text{ is the only eigenvector corresponding to } \lambda = 4.}$

Answer to Question 2.

(a) Because T stretches images by a factor of 2 horizontally, this means that anything on the x-axis is multiplied by a factor of 2. This means

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

So $\boxed{\begin{bmatrix}1\\0\end{bmatrix} \text{ is an eigenvector with eigenvalue } \lambda = 2.}$

Similarly, T stretches images by a factor of 3 vertically, so

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\3\end{bmatrix} = 3\begin{bmatrix}0\\1\end{bmatrix}$$

and $\boxed{\begin{bmatrix}0\\1\end{bmatrix} \text{ is an eigenvector with eigenvalue } \lambda = 3.}$

(b) Since T is a reflection across the x-axis, everything on the x-axis will remain unchanged under T. This means

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\0\end{bmatrix}$$

and $\boxed{\begin{bmatrix}1\\0\end{bmatrix} \text{ is an eigenvector with eigenvalue } \lambda = 1.}$

However, anything on the y-axis is flipped, so

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\-1\end{bmatrix} = -\begin{bmatrix}0\\1\end{bmatrix}$$

which means $\boxed{\begin{bmatrix}0\\1\end{bmatrix} \text{ is an eigenvector with eigenvalue } \lambda = -1.}$

(c) We can see that rotating any non-zero vector in $\mathbb{R}^2$ by $90°$ will *always* result in a vector that is not a scalar multiple of the original. Therefore $\boxed{T \text{ has no real eigenvalues.}}$

(It turns out that T does still have 2 eigenvalues, but they are imaginary.)

Answer to Question 3.

To compute the eigenvalues of A, we could find the roots of the characteristic equation as in Question 1.

However, since we already have the eigenvectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, we can just check for the eigenvectors by multiplying the original matrix A by $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, and seeing what the scalar multiple is:

$$A\boldsymbol{v}_1 = \begin{bmatrix}-3 & 12\\-2 & 7\end{bmatrix}\begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}3\\1\end{bmatrix} \qquad \implies \qquad \lambda_1 = 1$$

$$A\boldsymbol{v}_2 = \begin{bmatrix}-3 & 12\\-2 & 7\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}6\\3\end{bmatrix} \qquad \implies \qquad \lambda_2 = 3$$

Therefore, the diagonalization of A is:

$$\boxed{A = PDP^{-1} = \begin{bmatrix}3 & 2\\1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 3\end{bmatrix}\begin{bmatrix}3 & 2\\1 & 1\end{bmatrix}^{-1}}$$

Answer to Question 4.

(a) Starting with $a_0 = a_1 = 1$, we can compute the next few entries in the series as follows:

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \Longrightarrow \qquad a_2 = 2$$

$$\begin{bmatrix} a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \qquad \Longrightarrow \qquad a_3 = 3$$

$$\begin{bmatrix} a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \qquad \Longrightarrow \qquad a_4 = 5$$

$$\begin{bmatrix} a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix} \qquad \Longrightarrow \qquad a_5 = 8$$

The sequence $1, 1, 2, 3, 5, 8, \ldots$ is known as the **Fibonacci numbers.**

In fact, if we multiply out the equation:

$$\begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a_{n-1} \\ a_n \end{bmatrix} = \begin{bmatrix} a_n \\ a_n + a_{n-1} \end{bmatrix}$$

the first equation ($a_n = a_n$) is a tautology, and the second equation ($a_{n+1} = a_n + a_{n-1}$) is exactly the equation describing the Fibonacci numbers.

(b) To compute the eigenvalues of $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, we find the roots of the characteristic equation:

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} -\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix}\right) = 0$$

$$-\lambda(1-\lambda) - 1 = \lambda^2 - \lambda - 1 = 0$$

Using the quadratic formula,

$$\lambda = \frac{-(-1) \pm \sqrt{1 - 4(1)(-1)}}{2(1)}$$

and so the eigenvalues are:

$$\boxed{\lambda = \frac{1 \pm \sqrt{5}}{2}}$$

(c) We can check that these are eigenvectors as follows:

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1+\sqrt{5} \end{bmatrix} = \begin{bmatrix} 1+\sqrt{5} \\ 3+\sqrt{5} \end{bmatrix} = \left(\frac{1+\sqrt{5}}{2}\right) \begin{bmatrix} 2 \\ 1+\sqrt{5} \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1-\sqrt{5} \end{bmatrix} = \begin{bmatrix} 1-\sqrt{5} \\ 3-\sqrt{5} \end{bmatrix} = \left(\frac{1-\sqrt{5}}{2}\right) \begin{bmatrix} 2 \\ 1-\sqrt{5} \end{bmatrix}$$

(d) Given the eigenvectors and eigenvalues, we can diagonalize A as

$$A = PDP^{-1} = \begin{bmatrix} 2 & 2 \\ 1+\sqrt{5} & 1-\sqrt{5} \end{bmatrix} \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 1+\sqrt{5} & 1-\sqrt{5} \end{bmatrix}^{-1}$$

We can also compute that the inverse is:

$$\begin{bmatrix} 2 & 2 \\ 1+\sqrt{5} & 1-\sqrt{5} \end{bmatrix}^{-1} = \frac{1}{4\sqrt{5}} \begin{bmatrix} \sqrt{5}-1 & 2 \\ \sqrt{5}+1 & -2 \end{bmatrix}$$

(e) Combining our answers from the previous parts,

$$\begin{aligned} \begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \frac{1}{4\sqrt{5}} \begin{bmatrix} 2 & 2 \\ 1+\sqrt{5} & 1-\sqrt{5} \end{bmatrix} \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix}^n \begin{bmatrix} \sqrt{5}-1 & 2 \\ \sqrt{5}+1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \frac{1}{4\sqrt{5}} \begin{bmatrix} 2 & 2 \\ 1+\sqrt{5} & 1-\sqrt{5} \end{bmatrix} \begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^n \end{bmatrix} \begin{bmatrix} \sqrt{5}+1 \\ \sqrt{5}-1 \end{bmatrix} \\ &= \frac{1}{4\sqrt{5}} \begin{bmatrix} 2 & 2 \\ 1+\sqrt{5} & 1-\sqrt{5} \end{bmatrix} \begin{bmatrix} (\sqrt{5}+1)\left(\frac{1+\sqrt{5}}{2}\right)^n \\ (\sqrt{5}-1)\left(\frac{1-\sqrt{5}}{2}\right)^n \end{bmatrix} \end{aligned}$$

Because we only need the first component, this becomes:

$$a_n = \frac{1}{4\sqrt{5}} \left[(2\sqrt{5}+2)\left(\frac{1+\sqrt{5}}{2}\right)^n + (2\sqrt{5}-2)\left(\frac{1-\sqrt{5}}{2}\right)^n \right]$$

which we can simplify as:

$$\boxed{a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}}$$

(f) Substituting the approximate values into our result from the previous part,

$$a_n \approx \frac{1}{\sqrt{5}}(1.618...)^{n+1} - \frac{1}{\sqrt{5}}(-0.618...)^{n+1}$$

We can see that as n becomes large, the 1.618^{n+1} part dominates, and

$$a_n \approx \frac{1}{\sqrt{5}}(1.618...)^{n+1} \qquad \text{for large } n$$

Therefore,

$$\frac{a_{n+1}}{a_n} \approx \frac{(1.618...)^{n+2}}{(1.618...)^{n+1}} = 1.618...$$

in other words,

$$\boxed{\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1+\sqrt{5}}{2} \approx 1.618...}$$

Answer to Question 5.

(a) With back substitution, we solve for the value of one variable, "plug it in" to the next equation and use it to find the value of the next variable, and repeat.

For quasi upper triangular matrices, we can do something similar, except sometimes solving a 2×2 system instead of solving for a single variable.

For example, with the matrix: $\begin{bmatrix} \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & 0 & 0 & \times \end{bmatrix}$, we would:

- Solve for x_4
- Plug in x_4, then solve 2×2 system for x_2 and x_3
- Plug in x_2 and x_3, then solve for x_1

(b) To compare the eigenvalues, we can compare the two characteristic equations:

$$\det(T - \lambda I) \qquad \text{and} \qquad \det(UTU^T - \lambda I)$$

We can then rewrite the second one as:

$$\begin{aligned} \det(UTU^T - \lambda I) &= \det(UTU^T - \lambda UU^T) \\ &= \det(UTU^T - U\lambda IU^T) \\ &= \det(U(T - \lambda I)U^T) \\ &= \det(U)\det(T - \lambda I)\det(U^T) \end{aligned}$$

And since $U^T = U^{-1}$, we know that $\det(U^T) = \det(U^{-1}) = \frac{1}{\det(U)}$, so

$$\boxed{\det(UTU^T - \lambda I) = \det(T - \lambda I)}$$

and therefore A and T have the same characteristic equation, and thus the same eigenvalues.