Math 2940 Worksheet
The Carpet Ride Problem

Week 1
August 29, 2019

You are a young traveler, leaving home for the first time. Your parents want to help you on your journey, so just before your departure, they give you two gifts. Specifically, they give you two forms of transportation: a hover board and a magic carpet. Your parents inform you that both the hover board and the magic carpet have restrictions in how they operate:

We denote the restriction on the hover board's movement by the vector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.
By this we mean that if the hover board traveled "forward" for one hour, it would move along a "diagonal" path that would result in a displacement of 3 miles East and 1 mile North of its starting location. If it traveled "backward" for one hour, then it would result in a displacement of 3 miles West and 1 mile South of its starting location.

We denote the restriction on the magic carpet's movement by the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.
By this we mean that if the magic carpet traveled "forward" for one hour, it would move along a "diagonal" path that would result in a displacement of 1 mile East and 2 miles North of its starting location. If it traveled "backward" for one hour, then it would result in a displacement of 1 mile West and 2 miles South of its starting location.

Scenario One: The Maiden Voyage
Your Uncle Cramer suggests that your first adventure should be to go visit the wise man, Old Man Gauss. Uncle Cramer tells you that Old Man Gauss lives in a cabin that is 107 miles East and 64 miles North of your home.

Can you use the hover board and the magic carpet to get to Gauss's cabin?

If so, how? If it is not possible to get to the cabin with these modes of transportation, why is that the case?

Scenario Two: Hide and Seek

Old Man Gauss wants to move to a cabin in a different location. You are not sure whether Gauss is just trying to test your wits at finding him or if he actually wants to hide somewhere that you can't visit him.

Are there some locations that he can hide and you cannot reach him with these two modes of transportation?

Describe the places that you can reach using a combination of the hover board and the magic carpet and those you cannot. Try to specify these both geometrically and algebraically. Also, include a convincing argument supporting your answer.

Scenario Three: Getting Back Home

Suppose you are now in a three-dimensional world for the carpet ride problem, and you have three modes of transportation: $v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix}$, $v_3 = \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix}$

You are only allowed to use each mode of transportation **once** (in the forward or backward direction) for a fixed amount of time (c_1 on v_1, c_2 on v_2, c_3 on v_3). Find the amounts of time on each mode of transportation (c_1, c_2, and c_3, respectively) needed to go on a journey that starts and ends at home OR explain why it is not possible to do so.

- Is there more than one way to make a journey that meets the requirements described on the previous page? (In other words, are there different combinations of times you can spend on the modes of transportation so that you can get back home?) If so, how?

- Is there anywhere in this 3D world that Gauss could hide from you? If so, where? If not, why not?

Answer to Question 1. Suppose that we start at $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and travel on the hoverboard for t hours. Then our final location would be:

$$t \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3t \\ t \end{bmatrix}$$

Similarly, if we started at $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and traveled on the magic carpet for s hours, our final location would be:

$$s \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} s \\ 2s \end{bmatrix}$$

If we were to travel on the hoverboard for t hours, and then the magic carpet for s hours, we would find the final location by adding the two vectors:

$$t \begin{bmatrix} 3 \\ 1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

(Note: a lot of people in class used x and y here instead of t and s. That's certainly not wrong, but it can be a little misleading since you would be using x and y to measure *time* instead of vertical and horizontal positions.)

To reach Old Man Gauss' cabin, our final location would have to be $\begin{bmatrix} 107 \\ 64 \end{bmatrix}$. This gives us the vector equation:

$$t \begin{bmatrix} 3 \\ 1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 107 \\ 64 \end{bmatrix}$$

This is equivalent to a linear system of two equations in two unknowns:

$$\begin{aligned} 3t + s &= 107 \\ t + 2s &= 64 \end{aligned}$$

We'll learn a lot about solving linear systems of equations in the next few weeks, but one possible way to solve it is to take the first equation, and subtract three times the second equation to get:

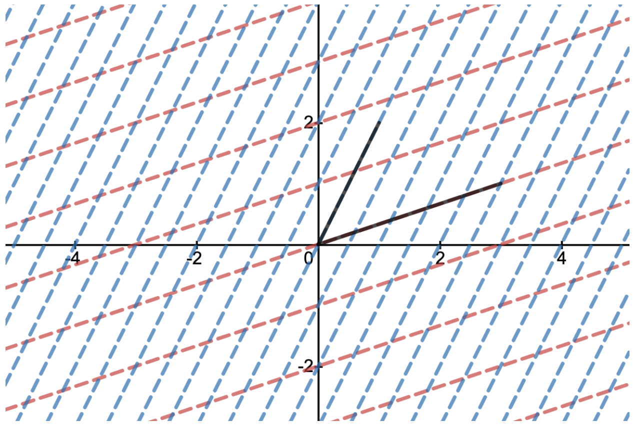
$$\begin{aligned} 3t + s - 3(t + 2s) &= 107 - 3(64) \\ -5s &= 107 - 192 \\ -5s &= -85 \\ s &= 17 \end{aligned}$$

Plugging $s = 17$ back into our original second equation ($3t + s = 107$), we get

$$\begin{aligned} 3t + 17 &= 107 \\ 3t &= 90 \\ t &= 30 \end{aligned}$$

Yes, you can get to Old Man Gauss' cabin by first riding the hover board for 30 hours, and then riding the magic carpet for 17 hours.

Answer to Question 2. It turns out that you actually can reach every location using the combination of the hover board and the magic carpet.
Geometrically, we can actually see that our two vectors $\begin{bmatrix}3\\1\end{bmatrix}$ and $\begin{bmatrix}1\\2\end{bmatrix}$ form a coordinate for a new "grid system" that looks like this:



where the red lines are parallel to $\begin{bmatrix}3\\1\end{bmatrix}$ and the blue lines are parallel to $\begin{bmatrix}1\\2\end{bmatrix}$, and the original vectors are solid black.
Note that this only works because our two vectors $\begin{bmatrix}3\\1\end{bmatrix}$ and $\begin{bmatrix}1\\2\end{bmatrix}$ are not parallel (i.e. they do not lie on the same line).

A more algebraic way of seeing this is to set up a linear system of equations as in Question 1, but now with unknowns for the right hand side. So if the cabin was at a location (x, y), then we need to find t and s such that

$$t\begin{bmatrix}3\\1\end{bmatrix} + s\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}x\\y\end{bmatrix}$$

This is equivalent to a linear system of two equations in two unknowns:

$$\begin{aligned}3t + s &= x\\ t + 2s &= y\end{aligned}$$

Like before, we subtract three times the second equation from the first equation to get:

$$\begin{aligned}-5s &= x - 3y\\ s &= \frac{3y - x}{5}\end{aligned}$$

Plugging this back into the original second equation,

$$\begin{aligned}t + 2\left(\frac{3y - x}{5}\right) &= y\\ t &= y - \left(\frac{6y - 2x}{5}\right)\\ t &= \frac{2x - y}{5}\end{aligned}$$

Since we can solve this system of equations like this for any values of x and y, this actually gives us exact formulas for how long to use each form of transport to end up at the location $\begin{bmatrix} x \\ y \end{bmatrix}$.

Answer to Question 3. If we use mode 1 for time c_1, mode 2 for time c_2, and mode 3 for time c_3, the our final location will be:

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix} + c_3 \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix}$$

To "get back home", our final location should be:

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix} + c_3 \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This is equivalent to the system of equations:

$$\begin{aligned} c_1 + 6c_2 + 4c_3 &= 0 \\ c_1 + 3c_2 + c_3 &= 0 \\ c_1 + 8c_2 + 6c_3 &= 0 \end{aligned} \tag{1}$$

We can see right away that $c_1 = c_2 = c_3 = 0$ solves this system of equations, but this doesn't actually answer the question of can we "get back home", since it would mean that we never left home in the first place.
So instead we want to find a *non-zero* solution to this system of equations. Let's start by trying to reduce the system of equations. If we subtract the first row from the second and third rows, we get

$$\begin{aligned} c_1 + 6c_2 + 4c_3 &= 0 \\ -3c_2 - 3c_3 &= 0 \\ 2c_2 + 2c_3 &= 0 \end{aligned}$$

Here we see that the last two equations are equivalent to each other, so we can simplify the system to

$$\begin{aligned} c_1 + 6c_2 + 4c_3 &= 0 \\ c_2 + c_3 &= 0 \end{aligned}$$

And subtracting four times the second equation from the first,

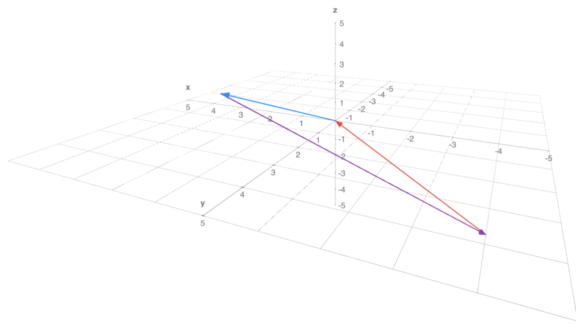
$$\begin{aligned} c_1 + 2c_2 &= 0 \\ c_2 + c_3 &= 0 \end{aligned} \tag{2}$$

Since we have three variables but only two distinct equations, this actually means that we have a *free* variable, and we can choose any value for that free variable to find a solution.

So, to find a way of getting back home, we could pick, for instance, $c_3 = 1$. Then to satisfy the second equation, we need $c_2 = -1$, and to satisfy the first equation we need to set $c_1 = 2$.

$\boxed{\text{Using } c_1 = 2,\ c_2 = -1, \text{ and } c_3 = 1 \text{ would be one possible way of getting back home.}}$

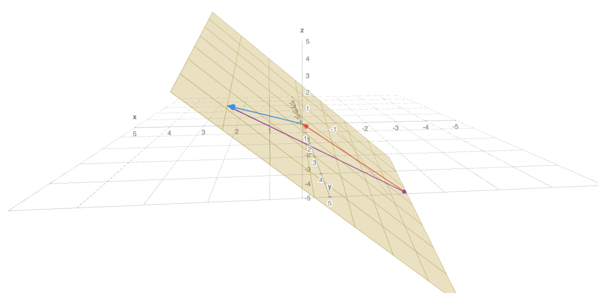
In 3D, this journey would look like:



You should be able to view an interactive version at this link: https://www.math3d.org/Po3s0VEq

We could also find many other solutions by picking different values for c_3, finding the corresponding values of c_1 and c_2 by solving system (2), and these would all give different ways of "getting back home".

Because we have a way of getting back home, we can think of $c_1v_1 + c_2v_2 + c_3v_3$ as forming a sort of triangle in three-dimensional space. Since this triangle is a flat two-dimensional surface, it is actually part of exactly one plane in three-dimensional space, and it follows that all three of our vectors are actually in this same plane.
The plane looks like this:



You should be able to view an interactive version at: https://www.math3d.org/80x88LQq

$\boxed{\text{Any trip we take using } v_1, v_2, \text{ and } v_3 \text{ will stay in this plane, and Gauss would be able to hide by choosing any point not in the same plane as } v_1, v_2, \text{ and } v_3.}$

What if we wanted to figure out exactly what this plane is? One way of doing so would be to find the normal vector describing the plane. If we wrote this vector as $\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$, then we want to check that it's perpendicular to v_1, v_2, and v_3 by using dot products:

$$\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0, \qquad \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix} = 0, \qquad \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix} = 0$$

(Technically we only need two of these equations, because we know all three vectors lie in the same plane. But using all three will still work.)
This gives us a *new* system of equations:

$$\begin{aligned} n_1 + n_2 + n_3 &= 0 \\ 6n_1 + 3n_2 + 8n_3 &= 0 \\ 4n_1 + n_2 + 6n_3 &= 0 \end{aligned}$$

Then, if we find a non-zero solution (n_1, n_2, n_3) to this system of equations (which I will cheekily leave up to the reader), the equation describing the plane spanned by v_1, v_2, and v_3 would be:

$$n_1x + n_2y + n_3z = 0$$