Math 2940 Notes
Matrix of a Linear Transformation

Week 4
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Linear Transformations

Much of linear algebra can be understood as the study of *linear transformations*. Here, *transformation* is just another fancy word for function. While *linear* just specifies that this function satisfies a few useful properties.
More precisely, we say that a function $T : \mathbb{R}^m \to \mathbb{R}^n$ is a *linear transformation* if it satisfies the two following properties for any scalar c and any vectors $\boldsymbol{u}$ and $\boldsymbol{v}$:

1. $T(c\boldsymbol{v}) = cT(\boldsymbol{v})$
2. $T(\boldsymbol{u} + \boldsymbol{v}) = T(\boldsymbol{u}) + T(\boldsymbol{v})$

Many common transformations turn out to be linear: reflections, rotations, contraction, dilation ...

Finding the Matrix of a Linear Transformation

One really useful fact about linear transformations is that every linear transformation can be represented as a matrix.
To see how, let's consider an m-dimensional vector $\boldsymbol{v}$, which we can write as

$$\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_m \end{bmatrix}$$

We can split this vector up in the following way:

$$\boldsymbol{v} = v_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + v_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + ... + v_m \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Each of these vectors with a 1 in the i-th entry and a 0 everywhere else is known as a *standard basis vector*, and is usually written as $\boldsymbol{e}_i$. So we can also write the above in shorthand as:

$$\boldsymbol{v} = v_1\boldsymbol{e}_1 + v_2\boldsymbol{e}_2 + v_3\boldsymbol{e}_3 + ... + v_m\boldsymbol{e}_m$$

Now, suppose we have a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$, and we want to figure out what $T(\boldsymbol{v})$ is. Then,

$$T(\boldsymbol{v}) = T\big(v_1\boldsymbol{e}_1 + v_2\boldsymbol{e}_2 + v_3\boldsymbol{e}_3 + ... + v_m\boldsymbol{e}_m\big)$$

Using the second property of linear transformations,

$$T(\boldsymbol{v}) = T\big(v_1\boldsymbol{e}_1\big) + T\big(v_2\boldsymbol{e}_2\big) + T\big(v_3\boldsymbol{e}_3\big) + ... + T\big(v_m\boldsymbol{e}_m\big)$$

Then using the first property of linear transformations,

$$T(\boldsymbol{v}) = v_1 T\big(\boldsymbol{e}_1\big) + v_2 T\big(\boldsymbol{e}_2\big) + v_3 T\big(\boldsymbol{e}_3\big) + ... + v_m T\big(\boldsymbol{e}_m\big)$$

It turns that we can write the right hand side as a matrix-vector product, which looks like:

$$T(\boldsymbol{v}) = \begin{bmatrix} T(\boldsymbol{e}_1) & T(\boldsymbol{e}_2) & T(\boldsymbol{e}_3) & ... & T(\boldsymbol{e}_m) \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_m \end{bmatrix}$$

$$T(\boldsymbol{v}) = \begin{bmatrix} T(\boldsymbol{e}_1) & T(\boldsymbol{e}_2) & T(\boldsymbol{e}_3) & ... & T(\boldsymbol{e}_m) \end{bmatrix} \cdot \boldsymbol{v}$$

where the i-th column of the matrix is given by the n-dimensional vector $T(\boldsymbol{e}_i)$.
So given any linear transformation T, we can find its matrix by calculating each $T(\boldsymbol{e}_i)$ and using it as the i-th column.