Math 2940 Notes
Linear Systems of Equations

Week 2
September 5th, 2019

Basic Terminology / Definitions

In this class, we'll be dealing a lot with *linear systems of equations*, such as the following:

$$\begin{aligned} x_1 + 2x_2 + 2x_3 &= 3 \\ x_1 + 2x_2 + x_3 &= 5 \\ 4x_1 + x_2 + 3x_3 &= 6 \end{aligned}$$

A useful shorthand way of writing linear systems is as an *augmented matrix*:

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 3 \\ 1 & 2 & 1 & 5 \\ 4 & 1 & 3 & 6 \end{array}\right]$$

(I find it helpful to use a vertical line like this to separate the coefficients from the right hand sides, but not everyone does this.)
To solve this system, we then can use *row operations*, which fall into the following categories:

- Multiply a row by a non-zero constant
- Switch two rows
- Add a multiple of one row to another row

Since each row of the augmented matrix represents an equation, these are all valid operations that don't change our solutions because we are either doing the same thing to both sides of the equation, or just re-ordering the equations.
The goal of using row operations is to get the matrix into either *echelon form* or *reduced echelon form*, which are sometimes referred to as *row echelon form* and *reduced row echelon form*, respectively.

Echelon Form

One example of a matrix in *echelon form* is:

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here, the black squares represent the leading non-zero entries in each row, and are known as the *pivots*. The $*$'s can be any value. In order to be in *echelon form*, the matrix has to obey the following rules:

- All nonzero rows are above any rows of all zeros.
- Each leading entry of a row is to the right of the leading entry of the row above it.

- All entries in a column below a leading entry are zeros.

Getting a matrix into echelon form is enough to tell us whether or not our linear system of equations has solutions or not. But if we want to actually figure out what those solutions are, we have to get it into *reduced echelon form*.

Reduced Echelon Form

Reduced echelon form is like *echelon form*, but with the following additional restrictions:

- The leading non-zero entry in each row is 1
- In every column that contains a leading 1, all other entries are 0

If the matrix above was brought into reduced echelon form, it would look like:

$$\begin{bmatrix} 1 & * & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Once we have a matrix in reduced echelon form, then we can easily figure out what the solutions to our linear system are.