Math 2930 Worksheet
Variation of Parameters
Free Vibrations

Week 8
March 15th, 2019

Variation of Parameters Formulas

For a second-order nonhomogeneous equation

$$y'' + p(t)y' + q(t)y = g(t)$$

with complementary solution

$$y_c = c_1 y_1 + c_2 y_2$$

the particular solution can be written as:

$$Y = u_1 y_1 + u_2 y_2$$

where

$$u_1'(t) = -\frac{y_2 g}{y_1 y_2' - y_1' y_2}, \qquad u_2'(t) = \frac{y_1 g}{y_1 y_2' - y_1' y_2}$$

Question 1. Use the method of variation of parameters to find the general solution of

$$y'' + 4y' + 4y = \frac{1}{t^2} e^{-2t}$$

Question 2. **(a)** Show that $y_1 = t^2$ and $y_2 = \frac{1}{t}$ are solutions to the homogeneous equation

$$t^2 y'' - 2y = 0$$

(b) Find the general solution of the non-homogeneous equation

$$t^2 y'' - 2y = 3t^2 - 1$$

using variation of parameters. (Remember, you already have the solution to the homogeneous problem from part **(a)**.)

Question 3. A mass of 1 kg stretches a spring 8 cm. The mass is first pushed upward, contracting the spring a distance of 2 cm, and then set in motion with a downward velocity of 60 cm/s. Assume that there is no damping and no external force is applied.

(To make things easier, you can assume that $g = 10m/s^2$*. Remember to watch your units!)*

(a) Find the position $u(t)$ of the mass at any time t.

(b) Determine the period, frequency, and the amplitude of the motion.

Question 4. The position of a certain undamped spring-mass system satisfies the initial value problem:

$$u'' + 2u = 0, \qquad u(0) = 0, \qquad u'(0) = 2$$

(a) Find the solution of this initial value problem

For the rest of this question, I would like you to use a graphing calculator (if you have one) or an online graphing tool (I personally recommend `desmos.com`). Laptops/calculators are strongly preferred, but you may use a smartphone if you have nothing else suitable.

(b) Plot u versus t and u' versus t on the same axes.

- What do the graphs of u and u' look like?
- What can you say about how the graphs of u and u' are related?

(c) Now plot u' versus u. By this I mean plot $u(t)$ and $u'(t)$ parametrically, with t as the parameter. This plot is known as a *phase plot*, and the uu' plane is called the *phase plane*.

(On Desmos, parametric plots should be formatted as $(x(t), y(t))$)

- What is the direction of motion on the phase plot as t increases?
- What does this graph tell you about the long-term behavior of $u(t)$?

(d) Repeat parts **(a)** and **(c)** , but now with:

$$u'' + u' + 2u = 0, \qquad u(0) = 0, \qquad u'(0) = 2$$

- How has the phase plot changed?
- How has the long-term behavior of $u(t)$ changed?

Question 5. It's actually possible to combine both reduction of order *and* variation of parameters at once to find the general solution of a non-homogenous second-order equation with only one part of the homogenous solution.

(a) Show that $y_1 = t$ is a solution of the corresponding homogenous equation for:

$$t^2 y'' - 2ty' + 2y = 4t^2 \tag{1}$$

(b) Let $y(t) = v(t)y_1(t) = vt$, and plug this into Equation (1) to find a equation for $v(t)$.

(c) Solve your equation from part **(b)** to find $v(t)$. Then multiply v by y_1 to get the general solution of (1). You should see that this method simultaneously finds both the second part of the homogenous solution y_2 *and* a particular solution Y.

Answer to Question 1. First, we want to solve the homogeneous problem to find $y_1(t)$ and $y_2(t)$. The homogeneous equation is:

$$y'' + 4y' + 4y = 0$$

Now, we find the roots of the characteristic equation:

$$\begin{aligned} r^2 + 4r + 4 &= 0 \\ (r+2)^2 &= 0 \\ r &= -2 \end{aligned}$$

so we have a repeated root of $r = -2$, and our two fundamental solutions are:

$$y_1(t) = e^{-2t}, \qquad y_2(t) = te^{-2t}$$

Taking derivatives,

$$y_1'(t) = -2e^{-2t}, \qquad y_2'(t) = e^{-2t} - 2te^{-2t}$$

From here, there are two different (but equivalent) ways of solving for u_1' and u_2'.

Method 1: Formulas for u_1' and u_2'

The first method is to use the formulas for u_1' and u_2' given at the top of the worksheet:

$$u_1'(t) = -\frac{y_2 g}{y_1 y_2' - y_1' y_2}, \qquad u_2'(t) = \frac{y_1 g}{y_1 y_2' - y_1' y_2}$$

(My apologies to the morning sections who had a typo in those formulas). These formulas both have the same denominator, which is known as the *Wronskian*, and the textbook writes using the notation:

$$W[y_1, y_2](t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$$

We aren't covering the section about the Wronskian this semester, so you don't need to know this, but I did want to explain the $W[y_1, y_w]$ notation that does pop up in the textbook section on variation of parameters.
Anyway, we can compute the Wronskian as:

$$\begin{aligned} W[y_1, y_2](t) &= y_1(t)y_2'(t) - y_1'(t)y_2(t) \\ &= \left(e^{-2t}\right)\left(e^{-2t} - 2te^{-2t}\right) - \left(-2te^{-2t}\right)\left(te^{-2t}\right) \\ &= e^{-4t} - 2te^{-4t} + 2te^{-4t} \\ &= e^{-4t} \end{aligned}$$

Now, we can compute u_1' using the formula:

$$\begin{aligned} u_1' &= -\frac{y_2 g}{W} = \frac{-\left(te^{-2t}\right)\left(\frac{1}{t^2}e^{-2t}\right)}{e^{-4t}} \\ u_1' &= \frac{-1}{t} \end{aligned}$$

which we can integrate to get

$$u_1 = -\ln(t) + C_1$$

Computing u_2' using the formula:

$$u_2' = \frac{y_1 g}{W} = \frac{\left(e^{-2t}\right)\left(\frac{1}{t^2}e^{-2t}\right)}{e^{-4t}}$$
$$u_2' = \frac{1}{t^2}$$

which we can integrate to get

$$u_2 = \frac{-1}{t} + C_2$$

Then our general solution is:

$$y = u_1 y_1 + u_2 y_2$$
$$y = \left(-\ln(t) + C_1\right)e^{-2t} + \left(\frac{-1}{t} + C_2\right)te^{-2t}$$
$$y = C_1 e^{-2t} + C_2 te^{-2t} - \ln(t)e^{-2t} - e^{-2t}$$

Since C_1 can be any constant, we can also simplify a little by combining the $-e^{-2t}$ term into the $C_1 e^{-2t}$ term to get:

$$\boxed{y = C_1 e^{-2t} + C_2 te^{-2t} - \ln(t)e^{-2t}}$$

Method 2: System of Equations

Instead of using the formulas for u_1' and u_2', we could also find them using the system of equations:

$$u_1' y_1 + u_2' y_2 = 0$$
$$u_1' y_1' + u_2' y_2' = g$$

substituting in y_1, y_2 and g for our specific problem,

$$u_1' e^{-2t} + u_2' te^{-2t} = 0$$
$$-2u_1' e^{-2t} + u_2'\left(e^{-2t} - 2te^{-2t}\right) = \frac{1}{t^2}e^{-2t}$$

Multiplying everything by e^{-2t},

$$u_1' + u_2' t = 0$$
$$-2u_1' + u_2'(1 - 2t) = \frac{1}{t^2}$$

The first equation gives us

$$u_1' = -u_2' t$$

substituting that into the second equation,

$$- - 2tu_2' + u_2' - 2tu_2' = \frac{1}{t^2}$$
$$u_2' = \frac{1}{t^2}$$

which we can integrate to get

$$u_2 = \frac{-1}{t} + C_2$$

Substituting back to find u_1', we get

$$u_1' = -tu_2' = \frac{-1}{t}$$

Integrating both sides,

$$u_1 = -\ln(t) + C_1$$

Then our general solution is:

$$\begin{aligned}
y &= u_1 y_1 + u_2 y_2 \\
y &= \left(-\ln(t) + C_1\right) e^{-2t} + \left(\frac{-1}{t} + C_2\right) te^{-2t} \\
y &= C_1 e^{-2t} + C_2 te^{-2t} - \ln(t)e^{-2t} - e^{-2t}
\end{aligned}$$

Since C_1 can be any constant, we can also simplify a little by combining the $-e^{-2t}$ term into the $C_1 e^{-2t}$ term to get:

$$\boxed{y = C_1 e^{-2t} + C_2 te^{-2t} - \ln(t)e^{-2t}}$$

Answer to Question 2. (a) First, let's show that $y_1 = t^2$ is a solution of the equation. Taking derivatives,

$$\begin{aligned}
y_1 &= t^2 \\
y_1' &= 2t \\
y_1'' &= 2
\end{aligned}$$

Plugging these into the differential equation,

$$\begin{aligned}
t^2 y'' - 2y &= 0 \\
t^2(2) - 2(t^2) &= 0 \\
0 &= 0
\end{aligned}$$

So $y_1 = t^2$ is a solution.
Now to show that $y_2 = \frac{1}{t}$ is a solution. Taking derivatives,

$$\begin{aligned}
y_2 &= \frac{1}{t} \\
y_2' &= \frac{-1}{t^2} \\
y_2'' &= \frac{2}{t^3}
\end{aligned}$$

Plugging these into the differential equation,

$$\begin{aligned}
t^2 y'' - 2y &= 0 \\
t^2\left(\frac{2}{t^3}\right) - 2\left(\frac{1}{t}\right) &= 0 \\
0 &= 0
\end{aligned}$$

So $y_2 = \frac{1}{t}$ is a solution.

(b) *Important*: The formulas I gave you are for the case when there is no coefficient in front of y''. Because of that, we should take our equation and divide everything by t^2 to get

$$y'' - \frac{2}{t^2}y = 3 - \frac{1}{t^2}$$

We already have from part **(a)** that

$$y_1 = t^2, \qquad y_2 = \frac{1}{t}$$

Taking derivatives,

$$y_1' = 2t, \qquad y_2' = \frac{-1}{t^2}$$

As with Question 1, there are two different (but equivalent) ways of proceeding here:

Method 1: Formulas for u_1' and u_2'

First, we calculate the Wronskian term (i.e. the denominator):

$$\begin{aligned} W[y_1, y_2](t) &= y_1 y_2' - y_1' y_2 \\ &= t^2\left(\frac{-1}{t^2}\right) - 2t\left(\frac{1}{t}\right) \\ &= -3 \end{aligned}$$

Then using our formula for u_1', we compute

$$\begin{aligned} u_1' &= \frac{-y_2 g}{W} = -\left(\frac{1}{t}\right)\left(\frac{1}{-3}\right)\left(3 - \frac{1}{t^2}\right) \\ u_1' &= \frac{1}{t} - \frac{1}{3t^3} \end{aligned}$$

Integrating to get u_1,

$$u_1 = \ln(t) + \frac{1}{6t^2} + C_1$$

Now, using our formula for u_2', we compute

$$\begin{aligned} u_2' &= \frac{y_1 g}{W} = \frac{t^2}{-3}\left(3 - \frac{1}{t^2}\right) \\ u_2' &= \frac{1}{3} - t^2 \end{aligned}$$

Integrating both sides,

$$u_2 = \frac{t}{3} - \frac{t^3}{3} + C_2$$

Putting this together, we get that our general solution is:

$$\begin{aligned} y &= u_1 y_1 + u_2 y_2 \\ y &= \left(\ln(t) + \frac{1}{6t^2} + C_1\right)t^2 + \left(\frac{t}{3} - \frac{t^3}{3} + C_2\right)\frac{1}{t}y \qquad = C_1 t^2 + \frac{C_2}{t} + \frac{1}{2} - \frac{t^2}{3} + t^2\ln(t) \end{aligned}$$

Since we can pull the $-\frac{t^2}{3}$ term into the $C_1 t^2$ term, an equivalent general solution is

$$\boxed{y = C_1 t^2 + \frac{C_2}{t} + \frac{1}{2} + t^2\ln(t)}$$

Method 2: System of Equations

This time I want to walk through the derivation of variation of parameters again, to show what happens with the t^2 term out in front. You don't need to show all of this work every time.
We want to find solutions of the form

$$y = u_1 y_1 + u_2 y_2$$

We'll want to plug our guess for y into the differential equation, so first let's take the derivative

$$y' = u_1' y_1 + u_2' y_2 + u_1 y_1' + u_2 y_2'$$

Now, since we have two functions u_1 and u_2 to solve for, but only one equation, we have an extra "degree of freedom" to work with. In order to simplify things, we will use one of those "degrees of freedom" to set:

$$u_1' y_1 + u_2' y_2 = 0 \tag{2}$$

This leaves us with

$$y' = u_1 y_1' + u_2 y_2'$$

Taking a derivative again, we get

$$y'' = u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2''$$

So plugging this into our differential equation,

$$t^2 y'' - 2y = t^2 \left(u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2'' \right) - 2\left(u_1 y_1 + u_2 y_2 \right) = 3t^2 - 1$$

Rearranging,

$$t^2 \left(u_1' y_1' + y_2' y_2' \right) + u_1 \left(t^2 y_1'' - 2y_1 \right) + u_2 \left(t^2 y_2'' - 2y_2 \right) = 3t^2 - 1$$

Now, since we know that y_1 and y_2 are solutions to the homogeneous equation, the u_1 and u_2 terms are multiplied by zero, so we are just left with:

$$t^2 \left(u_1' y_1' + u_2' y_2' \right) = 3t^2 - 1 \tag{3}$$

Equations (2) and (3) now form a system of equations that we can solve for u_1' and u_2'. Plugging in for y_1 and y_2, this gives the system of equations:

$$t^2 u_1' + \frac{1}{t} u_2' = 0$$
$$2t^3 u_1' - u_2' = 3t^2 - 1$$

Adding t times the first equation to the second, we get that

$$3t^3 u_1' = 3t^2 - 1$$

so

$$u_1' = \frac{1}{t} - \frac{1}{3t^3}$$

Integrating to get u_1,

$$u_1 = \ln(t) + \frac{1}{6t^2} + C_1$$

We can now substitute into the first equation to get:

$$u_2' = -t^3 u_1' = -t^3\left(\frac{1}{t} - \frac{1}{3t^3}\right) = \frac{1}{3} - t^2$$

Integrating to get u_2,

$$u_2 = \frac{t}{3} - \frac{t^3}{3} + C_2$$

Putting this together, we get that our general solution is:

$$y = u_1 y_1 + u_2 y_2$$

$$y = \left(\ln(t) + \frac{1}{6t^2} + C_1\right)t^2 + \left(\frac{t}{3} - \frac{t^3}{3} + C_2\right)\frac{1}{t}y \qquad = C_1 t^2 + \frac{C_2}{t} + \frac{1}{2} - \frac{t^2}{3} + t^2\ln(t)$$

Since we can pull the $-\frac{t^2}{3}$ term into the $C_1 t^2$ term, an equivalent general solution is

$$\boxed{y = C_1 t^2 + \frac{C_2}{t} + \frac{1}{2} + t^2\ln(t)}$$

Answer to Question 3. I'm going to do this problem symbolically, and only substitute in numbers at the very end. It's an old habit drilled into me by my former high school physics teacher. Personally, I much prefer doing it this way, but you don't have to do these problems the exact same way that I do.

(a) First, we want to calculate the spring constant k. At equilibrium, the spring is stretched a distance L, so there is a spring force acting upward on the mass with magnitude kL. There is also a gravitational force on the mass of mg. At equilibrium, these forces cancel each other out, so we have

$$mg - kL = 0$$

Which we can solve for k:

$$k = \frac{mg}{L}$$

Now that we have the spring constant k, we'll want to use this to get a differential equation. Let u be the displacement of the mass from equilibrium. Then the spring exerts a force on the mass of $-ku$ (negative since the force is in the opposite direction of the displacement). So Newton's second law says that:

$$mu'' = -ku$$

(Note that because we are measuring u as displacement *from equilibrium*, we don't have to worry about a gravity term here).

Using our value of $k = \frac{mg}{L}$, we can rewrite our differential equation for u as:

$$mu'' + \frac{mg}{L}u = 0$$

or equivalently,

$$u'' + \frac{g}{L}u = 0$$

The characteristic equation here is:

$$r^2 + \frac{g}{L} = 0$$

which (since g and L are positive), we can solve as follows:

$$r^2 = -\frac{g}{L}$$
$$r = \pm\sqrt{-\frac{g}{L}}$$
$$r = \pm\sqrt{\frac{g}{L}}i$$

Which tells us that

$$u(t) = C_1 \cos\left(\sqrt{\frac{g}{L}}t\right) + C_2 \sin\left(\sqrt{\frac{g}{L}}t\right)$$

Plugging in our initial position $u(0) = u_0$,

$$u(0) = C_1 = u_0$$

And plugging in our initial velocity $u'(0) = v_0$,

$$u'(0) = C_2\sqrt{\frac{g}{L}} = v_0$$
$$C_2 = v_0\sqrt{\frac{L}{g}}$$

So our position $u(t)$ is:

$$\boxed{u(t) = u_0 \cos\left(\sqrt{\frac{g}{L}}t\right) + v_0\sqrt{\frac{L}{g}} \sin\left(\sqrt{\frac{g}{L}}t\right)}$$

Substituting in the actual values here, I'm going to consistently do everything in meters. The problem statement gives us that:

$$u_0 = 0.02\text{m}, \qquad v_0 = -0.6\frac{\text{m}}{\text{s}}, \qquad g = 10\frac{\text{m}}{\text{s}^2}, \qquad L = 0.08\text{m}$$

Plugging this in, $\boxed{u(t) = 0.02\cos\left(5\sqrt{5}t\right) - \frac{0.6}{5\sqrt{5}}\sin\left(5\sqrt{5}t\right)}$

(b) From our solution for $u(t)$, we have that the frequency is:

$$\boxed{\omega = \sqrt{\frac{g}{L}} = 5\sqrt{5}\,\frac{\text{radians}}{\text{second}}}$$

The period is:

$$\boxed{T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}} = \frac{2\pi}{5\sqrt{5}}\text{ seconds}}$$

And the amplitude is:

$$\boxed{\sqrt{A^2 + B^2} = \sqrt{u_0^2 + \frac{v_0^2 L}{g}} = \sqrt{(0.02)^2 + \frac{(-0.6)^2(0.08)}{10}}\text{ meters}}$$

Answer to Question 4.

(a) We want to find the solution of

$$u'' + 2u = 0, \qquad u(0) = 0, \qquad u'(0) = 2$$

Solving for the roots of the characteristic polynomial,

$$\begin{aligned} r^2 + 2 &= 0 \\ r^2 &= -2 \\ r &= \pm\sqrt{2}i \end{aligned}$$

The corresponding general solution is

$$u(t) = c_1 \cos(\sqrt{2}t) + c_2 \sin(\sqrt{2}t)$$

Plugging in the initial condition $u(0) = 0$,

$$\begin{aligned} u(0) = c_1 \cos(0) + c_2 \sin(0) &= 0 \\ c_1 &= 0 \end{aligned}$$

Then plugging in the initial condition $u'(0) = 2$,

$$\begin{aligned} u'(t) &= -\sqrt{2}c_1 \sin(\sqrt{2}t) + \sqrt{2}c_2 \cos(\sqrt{2}t) \\ u'(0) = -\sqrt{2}c_1 \sin(0) + \sqrt{2}c_2 \cos(0) &= 2 \\ \sqrt{2}c_2 &= 2 \\ c_2 &= \sqrt{2} \end{aligned}$$

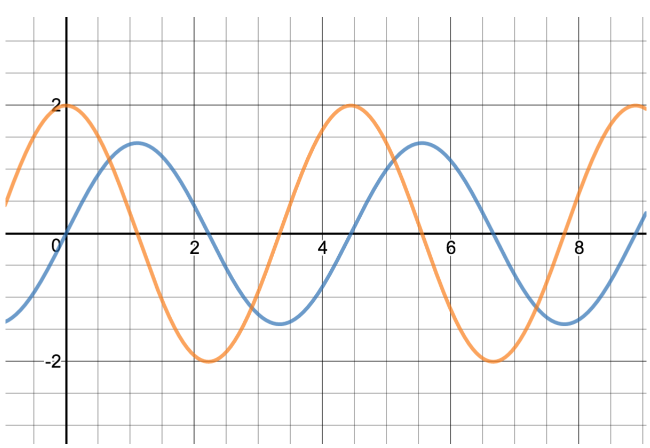
So the specific solution to this IVP is

$$\boxed{u(t) = \sqrt{2}\sin(\sqrt{2}t)}$$

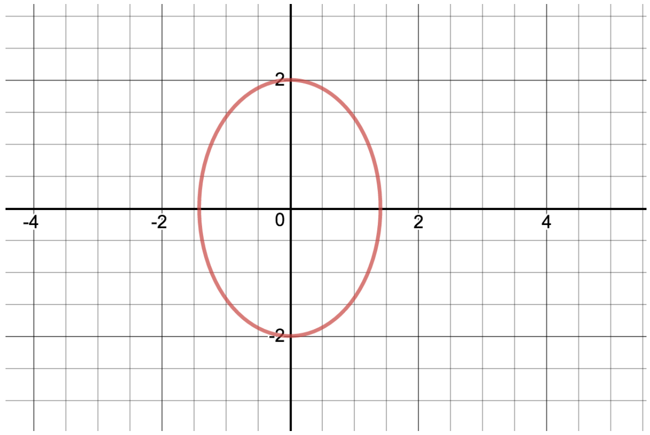
(b) The two functions we want to graph are

$$\begin{aligned} u(t) &= \sqrt{2}\sin(\sqrt{2}t) \\ u'(t) &= 2\cos(\sqrt{2}t) \end{aligned}$$

Graphed on the same axes, with u in blue and u' in orange



(c) If u and u' are plotted parametrically, the result is an ellipse:



Here, $u(t)$ and $u'(t)$ being periodic functions in t corresponds to this being a closed curve in the phase plane.
Increasing t corresponds to travelling clockwise around the ellipse.

(d) Now we want to repeat this with

$$u'' + u' + 2u = 0, \qquad u(0) = 0, \qquad u'(0) = 2$$

Solving for the roots of the characteristic equation,

$$r^2 + r + 2 = 0$$
$$r = \frac{-1 \pm \sqrt{1-8}}{2}$$
$$r = \frac{-1}{2} \pm \sqrt{7}i$$

This has corresponding general solution

$$u(t) = c_1 e^{-t/2} \cos(\sqrt{7}t) + c_2 e^{-t/2} \sin(\sqrt{7}t)$$

Plugging in $u(0) = 0$ gets us that $c_1 = 0$, so

$$u(t) = c_2 e^{-t/2} \sin(\sqrt{7}t/)$$

differentiating,

$$u'(t) = \frac{-c_2}{2} e^{-t/2} \sin(\sqrt{7}t) + \sqrt{7} c_2 e^{-t/2} \cos(\sqrt{7}t)$$

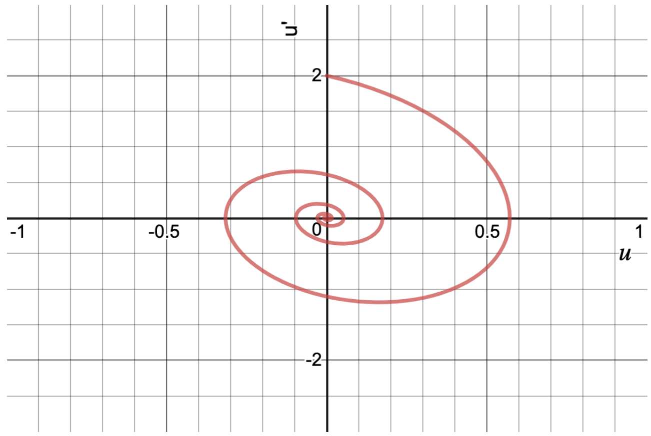
Plugging in $t = 0$,

$$u'(0) = \sqrt{7} c_2 = 2$$
$$c_2 = \frac{2}{\sqrt{7}}$$

So the solution to this IVP is

$$\boxed{u(t) = \frac{2}{\sqrt{7}} e^{-t/2} \sin(\sqrt{7}t)}$$

Then the graph of u' versus u in the phase plane is this inward spiral:



which tells us that the solution tends toward $u = 0, u' = 0$ as $t \to \infty$.

Answer to Question 5. (a) Let's check that $y_1 = t$ is a solution of the homogeneous equation. Taking derivatives,

$$\begin{aligned} y_1 &= t \\ y_1' &= 1 \\ y_1'' &= 0 \end{aligned}$$

Plugging this into the homogeneous equation,

$$\begin{aligned} t^2 y'' - 2ty' + 2y &= 0 \\ t^2(0) - 2t(1) + 2(t) &= 0 \\ 0 &= 0 \end{aligned}$$

So $y_1 = t$ is a solution of the homogeneous equation.

(b) Taking derivatives,

$$\begin{aligned} y &= vt \\ y' &= v't + v' \\ y'' &= v''t + 2v' \end{aligned}$$

Plugging this into the differential equation, we get

$$t^2\left(v''t + 2v'\right) - 2t\left(v't + v\right) + 2(vt) = 4t^2$$

After cancelling out terms, we are left with

$$t^3 v'' = 4t^2$$

which we can simplify to

$$\boxed{v'' = \frac{4}{t}}$$

(c) Normally, we would also have some v' terms, and so would have to solve a first-order equation in v'. However, in this case those all happened to cancel out, so to solve this we just have to integrate twice:

$$\begin{aligned} v'' &= \frac{4}{t} \\ v' &= 4\ln(t) + C_1 \\ v &= 4(t\ln(t) - t) + C_1 t + C_2 \end{aligned}$$

And so our general solution is:

$$y = vt = C_1 t^2 + C_2 t + 4t^2 \ln(t) - 4t^2$$

And we can simplify by combining the t^2 terms to get:

$$\boxed{y = C_1 t^2 + C_2 t + 4t^2 \ln(t)}$$

This method conveniently gives us both the complementary solution:

$$y_c = C_1 t^2 + C_2 t$$

and the particular solution:

$$Y = 4t^2 \ln(t)$$