Math 2930 Worksheet

Reduction of Order
Undetermined Coefficients

Week 7
March 8th, 2019

Question 1. Find the general solution of

$$y'' - y' + \frac{1}{4}y = e^{t/2}$$

Using the method of undetermined coefficients.

Question 2. Solve the following initial value problem

$$y'' + y' - 2y = 2t, \qquad y(0) = 0, \qquad y'(0) = 1$$

using the method of undetermined coefficients.

Question 3. For the following non-homogeneous equations, determine a suitable form for the particular solution $Y(t)$ is the method of undetermined coefficients is to be used.

(For this question, you do not need to actually solve for the coefficients!)

(a) $y'' + 2y' + 5y = 3te^{-t}\cos(2t) - 2te^{-2t}\cos(t)$

(b) $y'' + 3y' = 2t^4 + t^2e^{-3t} + \sin(3t)$

Question 4. Consider the following second-order differential equation:

$$t^2y'' - 2y = 0$$

Note: this equation does not have constant coefficients!

(a) Show that $y_1(t) = 3t^2$ is a solution of this equation.

(b) Using reduction of order, find the general solution to this differential equation.

Hint: Look for solutions of the form $y(t) = v(t)y_1(t)$, and use this to find an equation for $v(t)$.

Question 5. Consider the following second-order differential equation:

$$y'' - ty' + y = 0$$

(a) Show that $y_1(t) = t$ is a solution of this equation.

(b) Using reduction of order, find the general solution to this differential equation. (For this problem, it's OK to leave your answer in the form of an integral.)

Answer to Question 1. To find the general solution, we will split this problem into two parts. First, we solve for the homogeneous solution (which I call y_h). This is the solution to the homogeneous 2nd-order equation:

$$y'' - y' + \frac{1}{4}y = 0$$

Looking for solutions of the form $y = e^{rt}$, we get a characteristic polynomial of:

$$r^2 - r + \frac{1}{4} = 0$$

Using the quadratic formula,

$$\begin{aligned} r &= \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)\left(\frac{1}{4}\right)}}{2(1)} \\ r &= \frac{1 \pm \sqrt{1-1}}{2} = \frac{1 \pm 0}{2} \\ r &= \frac{1}{2}, \quad \frac{1}{2} \quad \text{(repeated)} \end{aligned}$$

Since we have a repeated root of $r = 1/2$, we get that the homogeneous solution is

$$y_h(t) = c_1 e^{t/2} + c_2 t e^{t/2}$$

Now we want to find the particular solution $Y(t)$ using the method of undetermined coefficients. Looking at the right hand side of $e^{t/2}$, we see that we should look for solutions with the term $e^{t/2}$ in them. However, because this root shows up *twice* in our characteristic equation, we will have to multiply the term by t^2 to make sure that our guess is not part of the homogeneous equation. Thus we will look for particular solutions of the form:

$$Y(t) = At^2 e^{t/2}$$

(*Note*: It was important that we found the homogeneous solution first, so that we would know to multiply by t^2 here. If we had just guessed $Y = Ae^{t/2}$, then everything would have cancelled out and we would have been stuck with $0 = e^{t/2}$, which clearly isn't going to help us solve for A.)

Taking derivatives (using the Product Rule),

$$\begin{aligned} Y' &= 2Ate^{t/2} + \frac{A}{2}t^2 e^{t/2} \\ Y'' &= 2Ae^{t/2} + Ate^{t/2} + Ate^{t/2} + \frac{A}{4}t^2 e^{t/2} \\ &= 2Ae^{t/2} + 2Ate^{t/2} + \frac{A}{4}t^2 e^{t/2} \end{aligned}$$

Plugging this back into the left hand side of our original equation, we calculate that

$$\begin{aligned} &Y''' - Y' + \frac{1}{4}Y \\ &= \left(2Ae^{t/2} + 2Ate^{t/2} + \frac{A}{4}t^2 e^{t/2}\right) - \left(2Ate^{t/2} + \frac{A}{2}t^2 e^{t/2}\right) + \frac{1}{4}\left(At^2 e^{t/2}\right) \\ &= 2Ae^{t/2} \end{aligned}$$

Setting this equal to the right hand side of the original equation,

$$2Ae^{t/2} = e^{t/2}$$

So clearly $A = \frac{1}{2}$, and our particular solution is:

$$Y = \frac{1}{2}t^2 e^{t/2}$$

Putting this together, the general solution $y = y_h + Y$ is:

$$\boxed{y = c_1 e^{t/2} + c_2 t e^{t/2} + \frac{1}{2}t^2 e^{t/2}}$$

Answer to Question 2. First we will find the solution y_h to the homogeneous equation

$$y'' + y' - 2y = 0$$

Looking for solutions of the form $y = e^{rt}$, we get the characteristic polynomial

$$r^2 + r - 2 = 0$$

which factors as

$$\begin{aligned} (r+2)(r-1) &= 0 \\ r = -2,& \quad +1 \end{aligned}$$

The corresponding homogeneous solution is

$$y + h = c_1 e^{-2t} + c_2 e^t$$

Now, for the particular solution $Y(t)$, since we have a linear polynomial on the right hand side, we will look for a $Y(t)$ that is a polynomial of the same degree (i.e. linear). In other words,

$$Y = At + B$$

Two things:

- We don't need to multiply by any additional powers of t as in Question 1, because $r = 0$ was not a root for the homogeneous part of the equation.
- Just looking for $Y = At$ won't work. We need the lower odd terms (here just $+B$) in order to cancel out terms that will show up when we differentiate.

Taking derivatives,

$$\begin{aligned} Y &= At + B \\ Y' &= A \\ Y'' &= 0 \end{aligned}$$

Plugging this into the left hand side of the original equation,

$$\begin{aligned} Y'' + Y' - 2Y &= A - 2(At + B) \\ &= (A - 2B) - 2At \end{aligned}$$

Setting this equal to the right hand side of the original equation,

$$2t = (A - 2B) - 2At$$

In order for this to hold for all values of t, we will need to compare like terms. We get one equation by setting the constant terms on both sides equal to each other:

$$0 = A - 2B$$

and another equation by setting the coefficients of the t terms on both sides equal to each other:

$$2 = -2A$$

We can solve this system of equations to get

$$A = -1, \qquad B = -\frac{1}{2}$$

which gives us a particular solution of

$$Y = -t - \frac{1}{2}$$

and a general solution of

$$y = c_1 e^{-2t} + c_2 e^{t} - t - \frac{1}{2}$$

Now to calculate the constants c_1 and c_2. (*Note*: we can't find c_1 and c_2 until *after* we have found the particular solution Y and included it in our general solution).
Taking the derivative of our general solution y with respect to t,

$$y'(t) = -2c_1 e^{-2t} + c_2 e^{t} - 1$$

Plugging in $t = 0$ and using the initial conditions, we get the following two equations for c_1 and c_2.

$$\begin{aligned} y(0) &= c_1 + c_2 - \frac{1}{2} = 0 \\ y'(0) &= -2c_1 + c_2 - 1 = 1 \end{aligned}$$

which we can solve to get

$$c_1 = \frac{-1}{2}, \qquad c_2 = 1$$

All together, the solution to the initial value problem is

$$\boxed{y = \frac{-1}{2} e^{-2t} + e^{t} - t - \frac{1}{2}}$$

Answer to Question 3. (a)
First, we need to solve the homogeneous part of the equation

$$y'' + 2y' + 5y = 0$$
$$r^{+}2r + 5 = 0$$

Using the quadratic formula,

$$r = \frac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i$$

The corresponding homogeneous solution is

$$y_h = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(t)$$

Now for the particular solution $Y(t)$. Since our equation is linear in y, we can split the particular solution into two parts:

- Y_1 corresponding to the $3te^{-t}\cos(2t)$ term.
- Y_2 corresponding to the $-2te^{-2t}\cos(t)$ term.

Then we can combine those two to get our particular solution:

$$Y = Y_1 + Y_2$$

First we will find Y_1. Based on the $3te^{-t}\cos(2t)$ term, we see that we will want a $A_0te^{-t}\cos(2t)$ term as part of Y_1.
In order to cancel out terms that will show up when we differentiate Y, we will also want the corresponding sine term $B_0te^{-2t}\sin(t)$. For the same reason, we will also want the lower order terms $A_1e^{-t}\cos(2t)$ and $B_1e^{-t}\sin(2t)$.
And since $e^{-t}\cos(2t)$ was part of the homogeneous solution, we will have to multiply all of these by a factor of t.
Together, that gives

$$Y_1(t) = (A_0t + A_1)te^{-t}\cos(2t) + (B_0t + B_1)e^{-t}\sin(2t)$$

Now for Y_2. Based on the $-2te^{-2t}\cos(t)$ term, we will start off with a $C_0te^{-2t}\cos(t)$ term.
As before, we will want the sine term $D_0te^{-2t}\sin(t)$ term and the lower order terms $C_1e^{-2t}\cos(t)$ and $D_1e^{-2t}\sin(t)$.
Since there are no corresponding roots in the homogeneous part of the equation, we do not have to multiply by a factor of t like we had to with Y_1.
Together, that gives

$$Y_2(t) = (C_0t + C_1)e^{-2t}\cos(t) + (D_0t + D_1)e^{-2t}\sin(t)$$

And our full particular solution is:

$$\boxed{\begin{aligned} Y(t) = Y_1(t) + Y_2(t) &= (A_0t + A_1)te^{-t}\cos(2t) + (B_0t + B_1)e^{-t}\sin(2t) \\ &\quad + (C_0t + C_1)e^{-2t}\cos(t) + (D_0t + D_1)e^{-2t}\sin(t) \end{aligned}}$$

Answer to Question 3(b) First, we need to solve the homogeneous part of the equation:

$$y'' + 3y' = 0$$

Looking for solutions of the form $y = e^{rt}$, we find the roots of the characteristic polynomial:

$$\begin{aligned} r^2 + 3r &= 0 \\ r(r+3) &= 0 \\ r = 0,\ -3 \end{aligned}$$

The corresponding homogeneous solution is

$$y_h = c_1 + c_2 e^{-3t}$$

Now, for finding the particular solution, we will split it up into three parts:

- Y_1 corresponding to $2t^4$
- Y_2 corresponding to $t^2 e^{-3t}$
- Y_3 corresponding to $\sin(3t)$

First, we'll find $Y_1(t)$. Since $2t^4$ is a fourth-degree polynomial, we will want Y_1 to be at least a fourth-degree polynomial as well. Moreover, because $r = 0$ was a part of the homogeneous solution, we will need to multiply Y_1 by a power of t. So Y_1 will be:

$$Y_1(t) = \left(A_1 t^4 + a_2 t^3 + A_3 t^2 + A_4 t + A_5\right) t$$

Now for $Y_2(t)$. Because of $t^2 e^{-3t}$, we will start with a term of $B_1 t^2 e - 3t$. To cancel out any lower-order terms that will show up when we differentiate, we're going to have to include $B_2 t e^{-3t}$ and $B_3 e^{-3t}$. And since $r = -3$ was part of the homogeneous solution, we will have to multiply all of this by t, giving us:

$$Y_2(t) = \left(B_1 t^2 + B_2 t + B_3\right) t e^{-3t}$$

And finally, $Y_3(t)$. Because of $\sin(3t)$, we will want a $C_1 \cos(3t)$ term, and the corresponding $C_2 \sin(3t)$ term. And since there was no $r = \pm 3i$ term in the homogeneous part of the problem, we don't need to multiply by a power of t, leaving:

$$Y_3(t) = C_1 \cos(3t) + C_2 \sin(3t)$$

Putting this all together, our full particular solution is:

$$\boxed{\begin{aligned} Y(t) = Y_1(t) + Y_2(t) + Y_3(t) = &\left(A_1 t^4 + a_2 t^3 + A_3 t^2 + A_4 t + A_5\right) t \\ &+ \left(B_1 t^2 + B_2 t + B_3\right) t e^{-3t} \\ &+ C_1 \cos(3t) + C_2 \sin(3t) \end{aligned}}$$

Answer to Question 4. (a) Taking derivatives,

$$\begin{aligned} y_1 &= 3t^2 \\ y_1' &= 6t \\ y_1'' &= 6 \end{aligned}$$

Plugging this into the original equation,

$$t^2y'' - 2y = t^2(6) - 2\left(3t^2\right) = 0$$

so this is a solution.

(b) Now we look for solutions of the form $y(t) = v(t)y_1(t)$. Using the y_1 from part **(a)** ,

$$y = vy_1 = 3vt^2$$

Taking derivatives (using the Product Rule),

$$\begin{aligned} y' &= 3v't^2 + 6vt \\ y'' &= 3v''t^2 + 12v't + 6v \end{aligned}$$

Plugging this into the original equation.

$$t^2y'' - 2y = t^2\left(3v''t^2 + 12v't + 6v\right) - 2\left(3vt^2\right) = 0$$

Cancelling terms,

$$3v''t^4 + 12v't^3 = 0$$

Which is a separable equation for $v'(t)$. (It's also linear, actually). In particular, we can separate variables as follows:

$$\begin{aligned} 3\frac{dv'}{dt}t^4 &= -12v't^3 \\ \frac{dv'}{v'} &= -\frac{4dt}{t} \end{aligned}$$

Integrating both sides,

$$\begin{aligned} \int \frac{1}{v'}dv' &= -4\frac{1}{t}dt \\ \ln(v') &= -4\ln(t) + C_1 \end{aligned}$$

Exponentiating both sides,

$$v' = C_1e^{-4\ln(t)} = \frac{C_1}{t^4}$$

Then integrating one more time, we get

$$v = \frac{C_1}{t^3} + C_2$$

So our general solution is

$$y = vy_1 = 3vt^2 = 3t^2\left(\frac{C_1}{t^3} + C_2\right)$$

$$\boxed{\frac{C_1}{t} + C_2t^2}$$

Answer to Question 5. (a) Taking derivatives,

$$\begin{aligned} y_1 &= t \\ y_1' &= 1 \\ y_1'' &= 0 \end{aligned}$$

Plugging this into the differential equation,

$$y'' - ty' + y = 0 - t(1) + t = 0$$

so this is a solution of this equation.

(b) We now look for a solution of the form

$$y = vy_1 = vt$$

Taking derivatives,

$$\begin{aligned} y &= vt \\ y' &= v't + v \\ y'' &= v''t + 2v' \end{aligned}$$

Plugging this back into the differential equation,

$$\begin{aligned} y'' - ty' + y &= \left(v''t + 2v'\right) - t\left(v't + v\right) + vt = 0 \\ &= v''t + \left(2 - t^2\right)v' = 0 \end{aligned}$$

which is a first-order separable equation for v'. Separating variables,

$$\begin{aligned} \frac{dv'}{dt}t &= \left(t^2 - 2\right)v' \\ \frac{dv'}{v'} &= \left(t - \frac{2}{t}\right)v' \end{aligned}$$

Integrating both sides,

$$\begin{aligned} \int \frac{1}{v'}dv' &= \int \left(t - \frac{2}{t}\right)dt \\ \ln(v') &= \frac{t^2}{2} - 2\ln(t) + C_1 \end{aligned}$$

Exponentiating both sides,

$$v' = \frac{C_1}{t^2}e^{t^2/2}$$

Integrating both sides again to get v,

$$v = C_1 \int \frac{1}{t^2}e^{t^2/2}dt + C_2$$

So our general solution $y = vt$ is:

$$\boxed{y = C_1 t \int \frac{1}{t^2}e^{t^2/2}dt + C_2 t}$$