Math 2930 Worksheet
2nd-order Homogeneous Equations

Week 6
March 1st, 2019

2nd-order Homogeneous Equations with Constant Coefficients

For second order equations of the form $ay'' + by' + cy = 0$, find the roots of the characteristic polynomial $ar^2 + br + c = 0$. Depending on the roots, your general solution will be:

Distinct real roots	Complex roots	Repeated real roots
$r_1, \quad r_2$	$a \pm bi$	$r_1, \quad r_1$
$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$	$y(t) = C_1 e^{at}\cos(bt) + C_2 e^{at}\sin(bt)$	$y(t) = C_1 e^{r_1 t} + C_2 t e^{r_1 t}$

Question 1. Consider the following differential equation

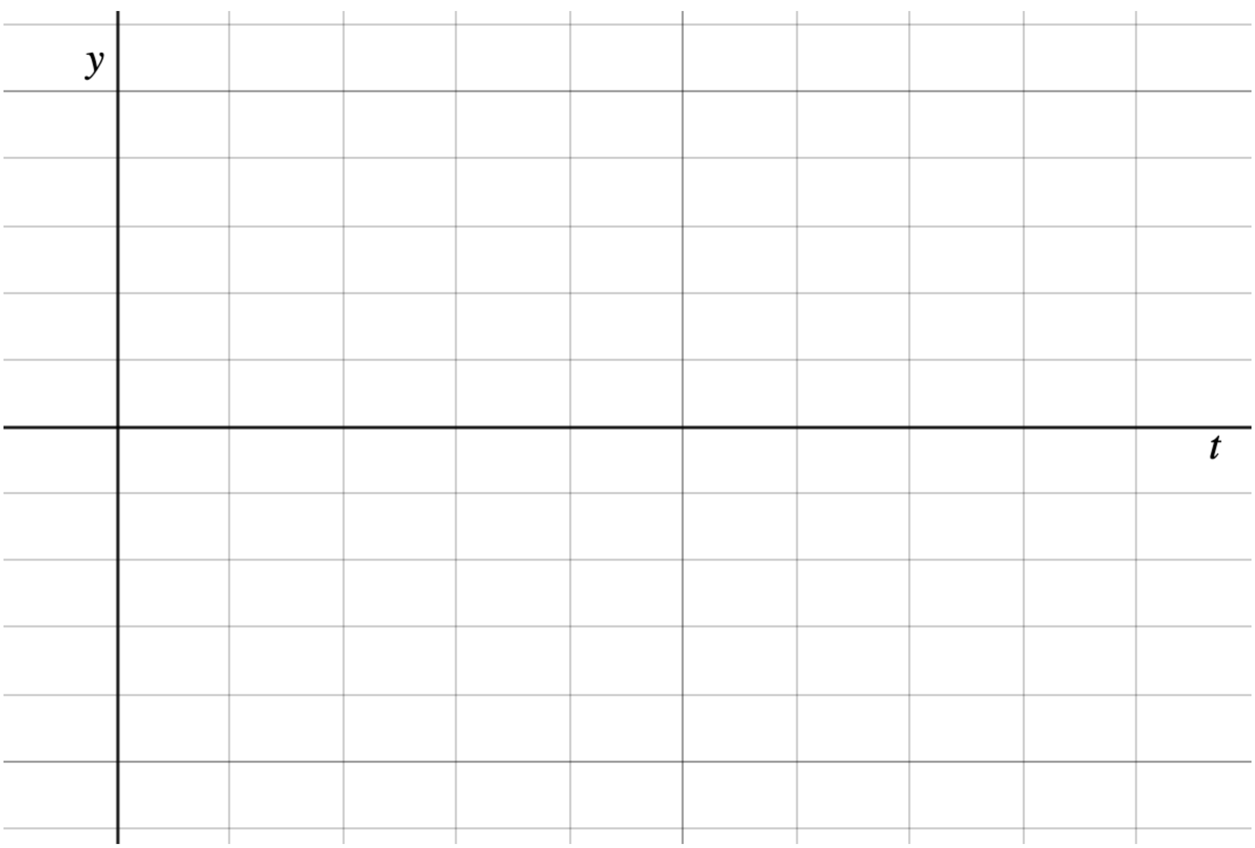
$$y'' + (3-\alpha)y' - 2(\alpha - 1)y = 0$$

Determine the values of α, if any, for which:
(a) all solutions tend to zero as $t \to \infty$
(b) all (nonzero) solutions become unbounded as $t \to \infty$.

Question 2. (a) Solve the initial value problem

$$y'' + 2y' + 2y = 0, \qquad y(0) = 1, \qquad y'(0) = -1$$

(b) Sketch the solution $y(t)$ on the axes below. How does y behave as $t \to \infty$?



Question 3. A wet sheet of paper was picked up off the ground by a Math 2930 student on the way to Collegetown. Much of the ink writing on the sheet was smudged by the water but clearly visible was "solution" followed by the mathematical expression

$$y(t) = e^{-2t}(\sin(t) + \cos(t))$$

Also visible were the words "second order constant coefficients" but the differential equation itself was smudged from the water except for

$$\cdots + 5y = 0$$

Other fragments of writing were also readable such as

$$y(0) = \ldots, \qquad y'(0) = \ldots$$

From this information

(a) Determine what the differential equation was.

(b) Determine what the initial conditions were.

Question 4. The position u of a certain spring-mass system satisfies the initial value problem

$$\frac{3}{2}u'' + ku = 0, \qquad u(0) = 2, \qquad u'(0) = v$$

(a) Solve for the motion $u(t)$ in terms of k and v.

(b) The period and amplitude of the resulting motion are observed to be π and 3, respectively. Determine the values of k and v.

(*Hint*: $A\cos(\omega t) + B\sin(\omega t)$ has amplitude $\sqrt{A^2 + B^2}$ and period $\frac{2\pi}{\omega}$)

Question 5. The differential equation:

$$t^2\frac{d^2y}{dt^2} - 4t\frac{dy}{dt} + 6y = 0$$

is an example of what is known as an *Euler equation.*
(Not to be confused with Euler's *formula* or Euler's *method.* If you haven't figured it out already, Leonhard Euler was kind of a big deal.)

(a) Upon first seeing equations like this, many students try to solve them in a way similar to solving equations with constant coefficients, which might look something like:

- "Guess" a solution of the form $y = e^{rt}$
- Write down the characteristic polynomial:

$$t^2r^2 - 4tr + 6 = 0$$

- Solve for r as a function of t using the quadratic formula
- Plug these two values of r back into $y = e^{rt}$, getting the two fundamental solutions as

$$y_1 = e^{r_1 t}, \qquad y_2 = e^{r_2 t}$$

But this actually doesn't work (you might find it helpful to check this yourself).
Can you explain why our method that works for constant coefficients does not correctly produce solutions here?

(b) Euler equations can be solved more generally using a change of variables. For the substitution $x = \ln(t)$, use the Chain Rule to show that:

$$\frac{dy}{dx} = t\frac{dy}{dt}, \qquad \frac{d^2y}{dx^2} = t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt}$$

(c) Show that this change of variables reduces the Euler equation for $y(t)$ into the following 2nd-order constant coefficient equation for $y(x)$:

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$

Then solve this equation for $y(x)$.

(d) Plug $x = \ln(t)$ back in, finding the general solution $y(t)$. (i.e. undo the change of variables).

Answer to Question 1. First, we plug in $y = e^{rt}$ to get the characteristic equation

$$r^2 + (3 - \alpha)r - 2(\alpha - 1) = 0$$

Then, using the quadratic formula to find r,

$$\begin{aligned}
r &= \frac{-(3-\alpha) \pm \sqrt{(3-\alpha)^2 - 4(-2)(\alpha - 1)}}{2} \\
r &= \frac{-3 + \alpha \pm \sqrt{9 - 6\alpha + \alpha^2 + 8\alpha - 8}}{2} \\
r &= \frac{-3 + \alpha \pm \sqrt{\alpha^2 + 2\alpha + 1}}{2} \\
r &= \frac{-3 + \alpha \pm (\alpha + 1)}{2} \\
r &= -2, \quad \text{and} \quad r = \alpha - 1
\end{aligned}$$

Therefore the general solution is

$$y = c_1 e^{-2t} + c_2 e^{(\alpha - 1)t}$$

(a) In order to guarantee that all solutions tend to zero as $t \to \infty$, we will need for both of these terms to be negative exponentials. The e^{-2t} term always tends to zero, but for *all* solutions to approach zero, we need

$$\boxed{\alpha - 1 < 0, \qquad \text{i.e.} \qquad \alpha < 1}$$

(b) Here we want to guarantee that *all* nonzero solutions become unbounded as $t \to \infty$. This part is actually trickier. Many students first think that the answer should be $\alpha > 1$, since that guarantees that the $e^{(\alpha-1)t}$ term in the solution is unbounded. But technically the question asks about *all* solutions, and even in this case there will still be some solutions that are bounded. Specifically, we wil have to include the corner cases where $c_2 = 0$, and those solutions are always bounded.

Put another way, for any value of α, we have that e^{-2t} is a solution, and this solution is always bounded. Therefore there are no values of α where *all* solutions are unbounded.

$$\boxed{\text{No such values of} \, \alpha}$$

Answer to Question 2.
(a) Looking for solutions of the form $y = e^{rt}$, we get the characteristic equation

$$r^2 + 2r + 2 = 0$$

Using the quadratic formula,

$$\begin{aligned}
r &= \frac{-2 \pm \sqrt{2^2 - r(2)(1)}}{2} \\
r &= \frac{-2 \pm \sqrt{-4}}{2} \\
r &= \frac{-2 \pm 2i}{2} = -1 \pm i
\end{aligned}$$

So the general solution is

$$y(t) = c_1 e^{-t} \cos(t) + c_2 e^{-t} \sin(t)$$

Taking the derivative, we get

$$y'(t) = -c_1 e^{-t} \cos(t) - c_! e^{-t} \sin(t) - c_2 e^{-t} \sin(t) + c_2 e^{-t} \cos(t)$$

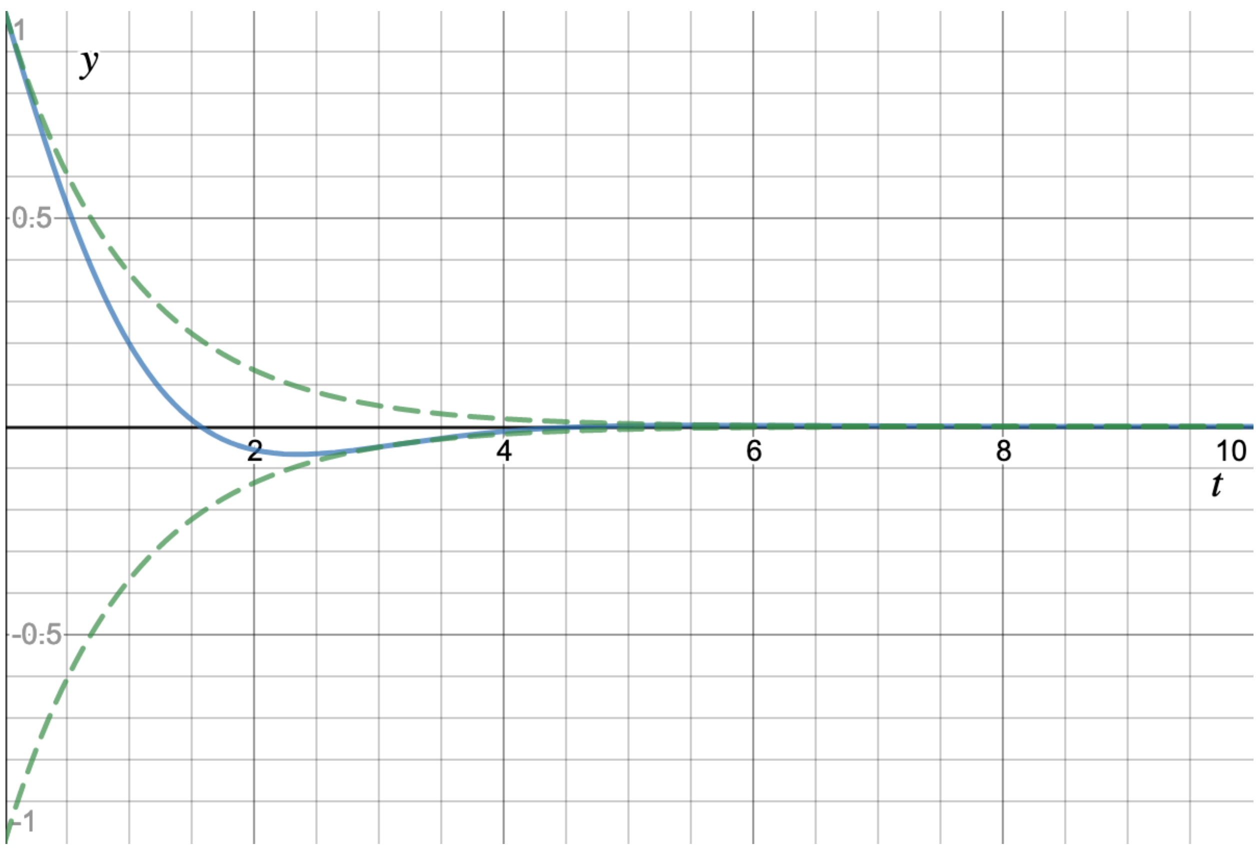
Plugging $t = 0$ into our equations for $y(t)$ and $y'(t)$, we get two equations for c_1 and c_2:

$$y(0) = c_1 + 0 = 1$$
$$y'(0) = -c_1 + c_2 = -1$$

So we get that $c_1 = 1$ and $c_2 = 0$. Thus our final solution is

$$\boxed{y = e^{-t} \cos(t)}$$

(b) To graph this function, we recall that as $\cos(t)$ oscillates between -1 and 1, similarly $e^{-t}\cos(t)$ oscillates between $-e^{-t}$ and e^{-t}. So to draw this graph by hand, we first sketch the graphs for e^{-t} and $-e^{-t}$ (shown as dashed green lines), and then draw a cosine function oscillating between those two limits (the blue graph)



Answer to Question 3.
(a) We have a second-order constant coefficient equation with solution

$$y(t) = e^{-2t}\sin(t) + e^{-2t}\cos(t)$$

Since these terms are the product of an exponential and a trig term, we can infer that our original secon-order equation must have had complex roots. More specifically, our general solution would have to have been:

$$y(t) = c_1 e^{-2t}\sin(t) + c_2 e^{-2t}\cos(t)$$

which means that the roots to the characteristic equation were

$$r = -2 \pm i$$

We also know that the original differential equation must have been of the form

$$ay'' + by' + 5y = 0$$

So the characteristic equation is

$$ar^2 + br + 5 = 0$$

Using the quadratic formula, we get r in terms of a and b:

$$r = \frac{-b \pm \sqrt{b^2 - 20a}}{2a}$$

Comparing our two different equations for r, we see that

$$-2 \pm i = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 20a}}{2a}$$

Setting the real parts equal to each other, and the imaginary parts equal to each other, we get two equations for a and b

$$-2 = \frac{-b}{2a}$$
$$i = \frac{\sqrt{b^2 - 20a}}{2a}$$

The first equation can be rearranged to get

$$b = 4a$$

Plugging that into the second equation,

$$i = \frac{\sqrt{16a^2 - 20a}}{2a}$$

Multiplying both sides by $2a$,

$$2ai = \sqrt{16a^2 - 20a}$$

Squaring both sides,

$$-4a^2 = 16a^2 - 20a$$

Rearranging,

$$a(a-1) = 0$$

Which has solutions $a = 0$ and $a = 1$. Since $a = 0$ would not be a second-order equation (it would also involve dividing by zero), we get that the solution is $a = 1$, and thus we also get $b = 4a = 4$. Putting that all together, the original differential equation was

$$\boxed{y'' + 4y' + 5y = 0}$$

(b) We have that the solution is

$$y(t) = e^{-2t}\sin(t) + e^{-2t}\cos(t)$$

Plugging in $t = 0$ on both sides, we get

$$y(0) = 1 + 0 = 0$$

If we also take the derivative of $y(t)$, we get

$$y'(t) = -2e^{-2t}\sin(t) + e^{-2t}\cos(t) - 2e^{-2t}\cos(t) - e^{-2t}\sin(t)$$

Plugging in $t = 0$ on both sides, we get

$$y'(0) = 0 + 1 - 2 - 0 = -1$$

So all together, the initial conditions were:

$$\boxed{y(0) = 1, \qquad y'(0) = -1}$$

Answer to Question 4. (a) The characteristic polynomial is:

$$\frac{3}{2}r^2 + k = 0$$

Solving for r,

$$r^2 = \frac{-2k}{3}$$
$$r = \pm\sqrt{\frac{-2k}{3}} = \pm\sqrt{\frac{2k}{3}}i$$

So the general solution is:

$$u(t) = C_1\cos\left(\sqrt{\frac{2k}{3}}t\right) + C_2\sin\left(\sqrt{\frac{2k}{3}}t\right)$$

Now we use the initial conditions to solve for C_1 and C_2.

$$u(0) = C_1(1) + C_2(0) = C_1 = 2$$

and

$$u'(0) = -\sqrt{\frac{2k}{3}}C_1(0) + \sqrt{\frac{2k}{3}}C_2(1) = \sqrt{\frac{2k}{3}}C_2(1) = v$$

(Don't forget the Chain Rule in calculating u')
So $C_2 = v\sqrt{\frac{3}{2k}}$. Putting these back into $u(t)$ we get:

$$\boxed{u(t) = 2\cos\left(\sqrt{\frac{2k}{3}}t\right) + v\sqrt{\frac{3}{2k}}\sin\left(\sqrt{\frac{2k}{3}}t\right)}$$

(b) A full period is the time it takes for the quantity inside the sin and cos functions to change from 0 to 2π. So we can solve for the period T as follows:

$$2\pi = \sqrt{\frac{2k}{3}}T$$
$$T = \frac{2\pi\sqrt{3}}{\sqrt{2k}} = \frac{\pi\sqrt{6}}{\sqrt{k}}$$

Since the problem gives us that the period T is π,

$$\pi = \frac{\pi\sqrt{6}}{\sqrt{k}}$$
$$k = 6$$

Now to solve for v.
Our amplitude is given by:

$$3 = \sqrt{2^2 + \left(v\sqrt{\frac{3}{2k}}\right)^2}$$

So we will solve this equation for v. Squaring both sides,

$$9 = 4 + \frac{3v^2}{2k}$$

Since we already figured out that $k = 6$,

$$5 = \frac{v^2}{4}$$
$$v^2 = 20$$
$$v = \pm\sqrt{20}$$

$$\boxed{v = \pm 2\sqrt{5}}$$

Answer to Question 5. (a) In order to get this characteristic polynomial, we assumed that r was a constant, and not a function of t.
If we then solve that characteristic polynomial using the quadratic formula, we are then saying that r is a function of t, making the derivation of the characteristic equation we just solved incorrect. If we properly thought of r as a function of t, then we would also have derivatives of r in our "characteristic equation" from the chain rule, and we would generally be no better off than we started.

(b) If we have $x = \ln(t)$, then we can calculate

$$\frac{dx}{dt} = \frac{1}{t}$$
$$\frac{dt}{dx} = t$$

With this, we can use the Chain Rule to relate derivatives in x to derivatives in t:

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}t$$

Then for the second derivative,

$$\frac{d^2y}{dx}^2 = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dx}\left[\frac{dy}{dt}t\right] = \frac{d}{dt}\left[\frac{dy}{dt}t\right] \cdot \frac{dt}{dx}$$

Then using the product rule,

$$\frac{d^2y}{dx^2} = \left[\frac{d^2y}{dt^2}t + \frac{dy}{dt}\right] \cdot \frac{dt}{dx} = \left[\frac{d^2y}{dt^2}t + \frac{dy}{dt}\right] t = t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt}$$

(c) Solving the characteristic polynomial,

$$r^2 - 5r + 6 = 0$$
$$(r-2)(r-3) = 0$$
$$r = 2, \quad 3$$

So the general solution is:

$$\boxed{y(x) = C_1e^{2x} + C_2e^{3x}}$$

(d) Plugging $x = \ln(t)$ into our answer from **(e)** ,

$$y(t) = C_1e^{2\ln(t)} + C_2e^{3\ln(t)}$$

which simplifies to

$$\boxed{y(t) = C_1t^2 + C_2t^3}$$