Math 2930 Worksheet
2nd-order Homogeneous Equations

Week 5
February 22nd, 2019

2nd-order Homogeneous Equations with Constant Coefficients

- For equations of the form
$$ay''(t) + by'(t) + cy(t) = 0$$
look for solutions of the form $y = e^{rt}$
- Find the roots r_1 and r_2 of the *characteristic polynomial*: $ar^2 + br + c = 0$
- General solution is $y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

Question 1. Solve the initial value problem

$$y'' + y' - 2y = 0, \qquad y(0) = 1, \qquad y'(0) = 1$$

Question 2. Can you find a differential equation whose general solution is

$$y = c_1 e^t + c_2 e^{-4t} \quad ?$$

Question 3. Given an initial value problem

$$ay'' + by' + cy = 0, \qquad y(0) = y_0, \qquad y'(0) = v_0$$

Suppose that for some r_1, r_2, the general solution is:

$$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

(a) In order to solve the initial value problem, C_1 and C_2 need to solve a system of two linear equations. What is that system of equations?

(b) Given $r_1 \neq r_2$, are you always guaranteed to be able to find C_1 and C_2 to solve the initial value problem?

If so, will C_1 and C_2 be unique?

Question 4. Consider the first-order differential equation

$$\frac{dy}{ds} = iy, \qquad y(0) = 1$$

where $i = \sqrt{-1}$

(a) Show that $y(s) = e^{is}$ is a solution of this initial value problem

(b) Show that $y(s) = \cos(s) + i\sin(s)$ is also a solution of this initial value problem.

(c) What does the uniqueness theorem imply about these two solutions?

(d) The answer to part c is known as *Euler's formula* (Note: this is very different from Euler's *method*, despite the confusingly similar names!).

Can you also prove Euler's formula using Taylor series? Recall that

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f''''(0)}{4!}x^4 + ...$$

(e) Replacing s with βt, and then multiplying by $e^{\alpha t}$, we get

$$e^{(\alpha+i\beta)t} = e^{\alpha t}\big(\cos(\beta t) + i\sin(\beta t)\big)$$

Can you find a similar formula for $e^{(\alpha-i\beta)t}$?

(f) Suppose you have two functions

$$A(t) = e^{\alpha t}\big(\cos(\beta t) + i\sin(\beta t)\big)$$
$$B(t) = e^{\alpha t}\big(\cos(\beta t) - i\sin(\beta t)\big)$$

Simplify the two following expressions:

$$x_1(t) = \frac{A(t) + B(t)}{2}$$

$$x_2(t) = i\frac{A(t) - B(t)}{2}$$

(g) What do you notice about $x_1(t)$ and $x_2(t)$ compared to $A(t)$ and $B(t)$?

(h) If $A(t)$ and $B(t)$ were solutions to a differential equation of the form

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0$$

would $x_1(t)$ and $x_2(t)$ be solutions too? How about $c_1x_1(t) + c_2x_2(t)$ for arbitrary constants c_1 and c_2?

Question 5. Find the general solution to the homogeneous differential equation

$$\frac{d^2x}{dt^2} + 25x = 0$$

You will find that your guess results in complex roots for the characteristic polynomial. Use your results from Question 4 to find the general solution to the differential equation.

Question 6. Solve the initial value problem

$$9y'' - 12y' + 4y = 0, \qquad y(0) = 2, \qquad y'(0) = -1$$

Answer to Question 1. Plugging in $y = e^{rt}$, we get the characteristic polynomial

$$r^{+}r - 2 = 0$$

This factors as

$$(r+2)(r-1) = 0$$

So we see that the two roots are $r = 1$ and $r = -2$. It follows that the general solution is:

$$y = c_1 e^t + c_2 e^{-2t}$$

The derivative of the general solution is:

$$y' = c_1 e^t - 2c_2 e^{-2t}$$

Plugging in the initial condition, we get two equations:

$$\begin{aligned} y(0) = c_1 + c_2 &= 1 \\ y'(0) = c_1 - 2c_2 &= 1 \end{aligned}$$

the solution to this system is $c_1 = 1$ and $c_2 = 0$. So plugging in those values of c_1 and c_2, we get that the solution to the intial value problem is

$$\boxed{y(t) = e^t}$$

Answer to Question 2. Since our general solution is

$$y = c_1 e^t + c_2 e^{-4t}$$

We want to find an equation where the roots of the characteristic polynomial are $r = 1$ and $r = -4$. One such polynomial is

$$\begin{aligned} (r-1)(r+4) &= 0 \\ r^2 + 3r - 4 &= 0 \end{aligned}$$

Which would come from the second-order equation

$$\boxed{y'' + 3y' - 4y = 0}$$

Answer to Question 3. (a)
Plugging in $t = 0$, we get that

$$y(0) = C_1 + C_2 = y_0$$

If instead we differentiate y to get

$$y'(t) = r_1 C_1 e^{r_1 t} + r_2 C_2 e^{r_2 t}$$

then plug in $t = 0$, we get

$$r_1 C_1 + r_2 C_2 = v_0$$

So all together, our system of equations is

$$\boxed{\begin{aligned} C_1 + C_2 &= y_0 \\ r_1 C_1 + r_2 C_2 &= v_0 \end{aligned}}$$

Or, written in matrix form:

$$\boxed{\begin{bmatrix} 1 & 1 \\ r_1 & r_2 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ v_0 \end{bmatrix}}$$

(b) The above system of linear equations has a unique solution if and only if the determinant is nonzero.
The determinant here is

$$(1)r_2 - (1)r_1 = r_2 - r_1$$

So given that $r_1 \neq r_2$, our determinant is never zero, and therefore

$$\boxed{\text{There always exists a unique pair } (C_1, C_2)}$$

Answer to Question 4. (a) Plugging $y = e^{is}$ into both sides of the equation

$$\begin{aligned} \frac{dy}{ds} &= iy \\ \frac{d}{ds}\left(e^{is}\right) &= i\left(e^{is}\right) \\ ie^{is} &= ie^{is} \end{aligned}$$

So $y = e^{is}$ is a solution of the differential equation. It also satisfies the initial condition because $y(0) = e^0 = 1$. Therefore,

$$\boxed{y(s) = e^{is} \text{ is a solution}}$$

(b) Plugging $y = \cos(t) + i\sin(t)$ into both sides of the equation,

$$\begin{aligned} \frac{dy}{ds} &= iy \\ \frac{d}{ds}\big(\cos(s) + i\sin(s)\big) &= i\big(\cos(s) + i\sin(s)\big) \\ -\sin(s) + i\cos(s) &= i\cos(s) - \sin(s) \end{aligned}$$

and since both sides are equal, that means $y = \cos(s) + i\sin(s)$ is a solution. It also satisfies the initial condition because

$$y(0) = \cos(0) + i\sin(0) = 1$$

Therefore,

$$\boxed{y(s) = \cos(s) + i\sin(s) \text{ is a solution}}$$

(c)

$$\frac{dy}{ds} = iy$$

is a linear first-order differential equation. So by Theorem 2.4.2 in the book, we know that there exists a unique solution to this differential equation for every initial condition.

Since we cannot have two distinct solutions to the initial value problem, it follows that our solutions in parts (a) and (b) must be the same. In other words,

$$\boxed{e^{is} = \cos(s) + i\sin(s)}$$

which is known as Euler's formula.

(d) We can compute the following Taylor series expansions:

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \\ \cos(x) &= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \dots \\ \sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \end{aligned}$$

Then plugging ix into the Taylor series for e^x, we get

$$\begin{aligned} e^{ix} &= 1 + ix + \frac{(ix)^2}{2} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \dots \\ e^{ix} &= 1 + ix - \frac{x^2}{2} - i\frac{x^3}{3!} + \frac{x^4}{4!} + \dots \end{aligned}$$

Separating the real and imaginary terms,

$$e^{ix} = \left(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \dots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)$$

We can see that the real part is actually just the Taylor series expansion of $\cos(x)$, and the imaginary component is actually the Taylor series expansion of $\sin(x)$. So this is another way of proving Euler's formula:

$$\boxed{e^{ix} = \cos(x) + i\sin(x)}$$

(e) First, we can write

$$e^{(\alpha - i\beta)t} = e^{\alpha t}e^{-i\beta t}$$

Then using Euler's formula on $e^{-i\beta t}$,

$$e^{(\alpha - i\beta)t} = e^{\alpha t}\big(\cos(-\beta t) + i\sin(-\beta t)\big)$$

$$\boxed{e^{(\alpha - i\beta)t} = e^{\alpha t}\big(\cos(\beta t) - i\sin(\beta t)\big)}$$

(f) We can simplify the two expressions as follows:

$$\begin{aligned} x_1(t) &= \frac{A(t) + B(t)}{2} \\ &= e^{\alpha t}\frac{(\cos(\beta t) + \cos(\beta t) + i\sin(\beta t) - i\sin(\beta t))}{2} \\ &= e^{\alpha t}\frac{2\cos(\beta t)}{2} \end{aligned}$$

$$\boxed{x_1(t) = e^{\alpha t}\cos(\beta t)}$$

$$\begin{aligned} x_1(t) &= i\frac{A(t) - B(t)}{2} \\ &= ie^{\alpha t}\frac{(\cos(\beta t) - \cos(\beta t) + i\sin(\beta t) + i\sin(\beta t))}{2} \\ &= ie^{\alpha t}\frac{2i\sin(\beta t)}{2} \end{aligned}$$

$$\boxed{x_1(t) = -e^{\alpha t}\sin(\beta t)}$$

(g) $x_1(t)$ and $x_2(t)$ are real-valued, while $A(t)$ and $B(t)$ are real numbers.

(h) Since this is a homogeneous linear equation, we get that linear combinations of solutions are also solutions. Therefore $x_1(t)$, $x_2(t)$, and $c_1x_1(t) + c_2x_2(t)$ are all solutions.

Answer to Question 5. We have the homogeneous differential equation

$$x'' + 25x = 0$$

Using our guess of $x = e^{rt}$, we find the characteristic polynomial to be

$$r^2 + 25 = 0$$

which has roots

$$r = \pm 5i$$

Based on our answer to Question 4, but with $\alpha = 0$, and $\beta = 5$, our general solution is:

$$\boxed{x = c_1\cos(5t) + c_2\sin(5t)}$$

Answer to Question 6. Our differential equation is

$$9y'' - 12y' + 4y = 0$$

Using our guess of $y = e^{rt}$, we find the characteristic polynomial to be

$$9r^2 - 12r + 4 = 0$$

which factors as

$$(3r - 2)^2 = 0$$

so therefore we have a *repeated root* of $r = \frac{2}{3}$.
Since this is a repeated root (see Section 3.4 of the book), our general solution is

$$y = c_1e^{2t/3} + c_2te^{2t/3}$$

And our derivative is

$$y' = \frac{2c_1}{3}e^{2t/3} + c_2e^{2t/3} + \frac{2c_2t}{3}e^{2t/3}$$

Plugging in $t = 0$, we get that

$$y(0) = c_1 = 2$$
$$y'(0) = \frac{2}{3}c_1 + c_2 = -1$$

Plugging $c_1 = 2$ into the second equation,

$$\frac{4}{3} + c_2 = -1$$

which we can solve to get $c_2 = -\frac{7}{3}$. Therefore the solution to the initial value problem is

$$\boxed{y(t) = 2e^{2t/3} - \frac{7t}{3}e^{2t/3}}$$