Math 2930 Worksheet

Prelim 1 Review

Week 6

September 28, 2017

Question 1.

Consider the following differential equation

$$y'' - (2\alpha - 1)y' + \alpha(\alpha - 1)y = 0$$

where α is a real parameter. Determine the values of α, if any, for which

(a) all solutions of the above equation tend to zero as $t \to \infty$;

(b) all (nonzero) solutions become unbounded as $t \to \infty$.

Question 2.
Find the general solution for each of the following differential equations:
(a)

$$y' = e^{2x} + y - 1$$

(b)

$$(\cos x + \ln y)\,dx + \left(\frac{x}{y} + e^{y}\right) dy = 0$$

Question 3. Consider the equation

$$\frac{dy}{dt} = y$$
$$y(0) = 1$$

(a) Find the analytical solution for $y(t)$ for the given initial condition

(b) If instead we solve the equation using the forward Euler's method, with a step size of h, write down the first 2 iterations. Express your answers in terms of h.

(c) Based on **(b)** , write down the expression after n-th iteration.

(d) Let n be the number of steps over the interval $[0, t]$, with $n = t/h$, show that in the limit as $h \to 0$, and $n \to \infty$, the numerical answer given by Euler's method converges to the analytical solution that you found in part **(a)** .

Question 4. The amount of C-14 in a fossil decays in time, and this decay can be described by a linear model based on the assumption that the rate of decay is proportional to the amount of substance remaining at time t.

(a) Write down and solve the differential equation describing the radioactive decay of C-14.

(b) Use the solution to estimate the age of a fossil which contains 0.1% of its original amount of C-14. Assume that the half-life of C-14 is 5700 years (The half-life of a radioactive substance is the time it takes for one-half of the atoms in an initial amount of the substance to disintegrate). In your estimates, you may take $\ln 10/\ln 2 \approx 3.32$.

Question 5. Consider the differential equation: $y' = y - y^3$

(a) Find the equilibrium solutions and determine which of these solutions are asymptotically stable, semistable, and unstable.

(b) Draw the phase line and sketch several solution curves in the ty-plane for $t > 0$.

(c) Assuming that the solution $y(t)$ has the initial value $y(0) = -\frac{1}{2}$, compute the limit of $y(t)$ as $t \to +\infty$.

Question 6. When a raindrop falls from a cloud, it evaporates (i.e. loses its mass through evaporation). Assume that the raindrop evaporates in such a manner that its shape remains spherical, and the rate of evaporation is proportional to its surface area.

(a) Show that the rate at which the radius r of the raindrop decreases is a constant c inversely proportional to the density ρ of water.

(b) A model for the velocity $v(t)$ of the raindrop is given by the differential equation

$$\frac{dv}{dt} + \frac{3c}{(ct + r_0)} v = g$$

Here $c < 0$ is the constant introduced in Part **(a)** , r_0 is the radius of the raindrop at $t = 0$, and g is the acceleration due to gravity. Assuming that the raindrop falls from rest, solve the equation for $v(t)$. Explain why this model is physically meaningful only if $0 \leq t < -r_0/c$

Answer to Question 1. To find the general solution of this equation, we look at the characteristic polynomial:

$$r^2 - (2\alpha - 1)r + \alpha(\alpha - 1) = 0$$

this factors as

$$(r - \alpha)(r - \alpha + 1) = 0$$

(If you didn't see that it factors, you could just use the quadratic equation instead.) The roots of this equation are

$$r = \alpha, \quad \alpha - 1$$

so the general solution is

$$y(t) = C_1 e^{\alpha t} + C_2 e^{(\alpha-1)t}$$

(a) In order for *all* solutions to tend to zero as $t \to \infty$, we would need that both of the fundamental solutions $e^{\alpha t}$ and $e^{(\alpha-1)t}$ go to zero. This happens when both α and $\alpha - 1$ are negative, which is the range:

$$\boxed{\alpha < 0}$$

(b) In order for *all* solutions to become unbounded as $t \to \infty$, this would require that both of the fundamental solutions $e^{\alpha t}$ and $e^{(\alpha-1)t}$ become unbounded. This happens when both α and $\alpha - 1$ are positive, which is the range:

$$\boxed{\alpha > 1}$$

Answer to Question 2.
(a) This is a first-order linear ODE, so we can solve it using integrating factors. To do that, we write it as:

$$y' - y = e^{2x} - 1$$

We want to find a function $\mu(x)$, so that when we multiply the entire equation by μ, the left hand side becomes something that looks like a product rule. So we want the left hand side of

$$\mu y' - \mu y = \mu e^{2x} - \mu$$

to look like a product rule. This will happen when

$$\frac{d\mu}{dt} = -\mu$$

which has as a solution

$$\mu(x) = e^{-x}$$

So our equation becomes

$$\begin{aligned} e^{-x}y' - e^{-x}y &= e^x - e^{-x} \\ \left(e^{-x}y\right)' &= e^x - e^{-x} \end{aligned}$$

Integrating both sides,

$$e^{-x}y = e^x + e^{-x} + C$$

then multiplying both sides by e^x, we get the solution

$$\boxed{y = e^{2x} + 1 + Ce^x}$$

(b) This equation is in the form $M\,dx + N\,dy = 0$, and is not separable, so we'll try checking if it is an exact equation.
To check if it's exact, we compute:

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left[\cos x + \ln y\right] = \frac{1}{y}$$

and

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left[\frac{x}{y} + e^y\right] = \frac{1}{y}$$

Since these two partial derivatives are equal, the equation is exact.
So we want to look for a function $\Phi(x, y)$, whose partial derivatives are

$$\frac{\partial \Phi}{\partial x} = M = \cos x + \ln y$$
$$\frac{\partial \Phi}{\partial y} = N = \frac{x}{y} + e^y$$

Integrating both of these functions, we get

$$\Phi = \sin x + x \ln y + f(y)$$
$$\Phi = x \ln y + e^y + g(x)$$

for some functions f and g. Comparing these two equations, we see that $f(y) = e^y$ and $g(x) = \sin x$, so the general solution is

$$\Phi(x, y) = \boxed{\sin x + x \ln y + e^y = C}$$

Answer to Question 3.
(a) We have the equation

$$\frac{dy}{dt} = y$$

This is separable, so we solve it as follows:

$$\int \frac{dy}{y} = \int dt$$
$$\ln(y) = t + C$$
$$y = Ce^t$$

plugging in the initial condition $y(0) = 1$, we get that $C = 1$, resulting in an analytical solution of

$$\boxed{y(t) = e^t}$$

(b) With Euler's method, we start with the initial condition:

$$y_0 = \boxed{1}$$

The first step gives

$$y_1 = y_0 + hf(y_0) = y_0 + hy_0 = (1+h)y_0 = \boxed{1+h}$$

then the second step gives

$$y_2 = y_1 + hf(y_1) = y_1 + h(y_1) = (1+h)y_1 = (1+h)(1+h) = \boxed{(1+h)^2}$$

(c) From **(b)** we see that there is a pattern that

$$y_{n+1} = (1+h)y_n$$

resulting in the formula:

$$\boxed{y_n = (1+h)^n}$$

(d) Taking our answer from **(c)** and replacing n with t/h, we have

$$y_n = (1+h)^{t/h}$$

Then we are interested in taking the limit as $h \to 0$

$$\lim_{h\to 0} y_n = \lim_{h\to 0}(1+h)^{t/h}$$

To figure out this limit, it helps to take the logarithm of both sides,

$$\lim_{h\to 0} \ln(y_n) = \lim_{h\to 0} \ln\left[(1+h)^{t/h}\right] = \lim_{h\to 0} \frac{t\ln(1+h)}{h}$$

this can then be evaluated using L'hôpital's rule,

$$\lim_{h\to 0} \ln(y_n) = \lim_{h\to 0} \frac{t\frac{1}{1+h}}{1} = t$$

Then since

$$\lim_{h\to 0} \ln(y_n) = t$$

we have

$$\boxed{\lim_{h\to 0} y_n = e^t}$$

matching our solution from part **(a)** .
Note: you can also do this problem by taking the limit as $n \to \infty$ instead of $h \to 0$ in a very similar way, but the details are a little trickier.

Answer to Question 4. (a) Let's use the function $Q(t)$ to represent the amount of C-14 as a function of time t in years.
Then it's rate of decay is $\frac{dQ}{dt}$, so this being proportional to the amount of C-14 remaining at time t means that this decay is described by the differential equation

$$\boxed{\frac{dQ}{dt} = rQ}$$

for some constant r.
This is a separable equation that we can solve as follows:

$$\begin{aligned} \int \frac{dQ}{Q} &= \int r\,dt \\ \ln Q &= rt + C \\ Q &= Ce^{rt} \end{aligned}$$

where the constant C is given by the initial amount Q_0 of the substance, so our solution is

$$\boxed{Q(t) = Q_0 e^{rt}}$$

(b) First, we can use the half-life to determine our constant r. We know that after 5700 years, we have half of the initial amount of C-14 remaining. In terms of equations, this means

$$Q(5700) = \frac{1}{2} Q_0$$

so plugging this into our solution from part **(a)** ,

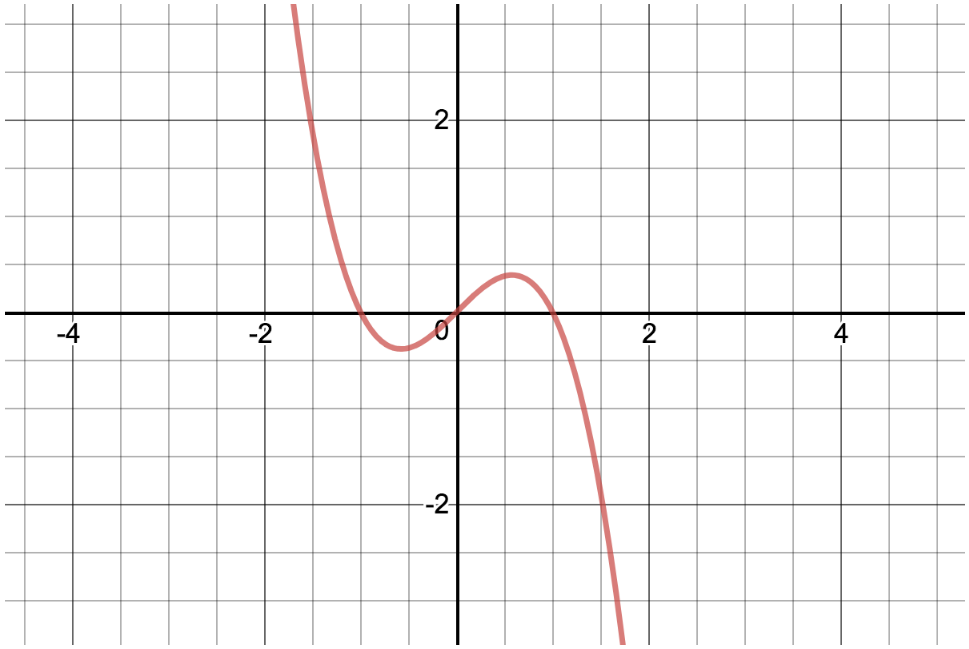
$$\begin{aligned} Q(5700) = Q_0 e^{5700r} &= \frac{1}{2} Q_0 \\ e^{5700r} &= \frac{1}{2} \\ 5700r &= \ln(1/2) = -\ln(2) \\ r &= \frac{-\ln(2)}{5700} \end{aligned}$$

So if we want to find the time τ after which only 0.1% of the substance is remaining, this means we have

$$\begin{aligned} Q(\tau) = Q_0 e^{r\tau} &= 0.001 Q_0 \\ e^{r\tau} &= 0.001 \\ r\tau &= \ln(0.001) = \ln(10^{-3}) = -3\ln(10) \\ \tau &= \frac{-3\ln(10)}{r} = \frac{-3\ln(10)5700}{\ln(2)} \approx -3(3.32)5700 \end{aligned}$$

$$\boxed{\tau = 56,772 \text{ years}}$$

Answer to Question 5.
(a) Since this is an autonomous equation, we'll first look at the graph of $\frac{dy}{dt}$ (vertical axis) vs y (horizontal axis):



The equilibria occur where this graph crosses the horizontal axis, these are the values of y where $\frac{dy}{dt} = 0$. Solving for them,

$$\frac{dy}{dt} = y - y^3 = 0$$
$$y(1 - y^2) = y(1 - y)(1 + y) = 0$$
$$y = 0, \quad 1, \quad -1$$

To figure out whether these equilibrium solutions are stable, unstable, or semistable, we will look at the sign of $\frac{dy}{dt}$ nearby.
For $y = -1$, we see that for values of y less than -1, $\frac{dy}{dt}$ if positive, so solutions are increasing. For values of y slightly greater than -1, we see $\frac{dy}{dt}$ is negative, so solutions are decreasing. Since solutions below $y = -1$ are increasing and solutions above are decreasing, we get that

$$\boxed{y = -1 \text{ is a } \textit{stable} \text{ equilibrium}}$$

For $y = 0$, solutions slightly below are decreasing and solutions slightly above are increasing, so

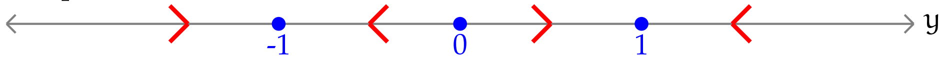
$$\boxed{y = 0 \text{ is an } \textit{unstable} \text{ equilibrium}}$$

For $y = 1$, solutions slightly below are increasing and solutions slightly above are decreasing, so

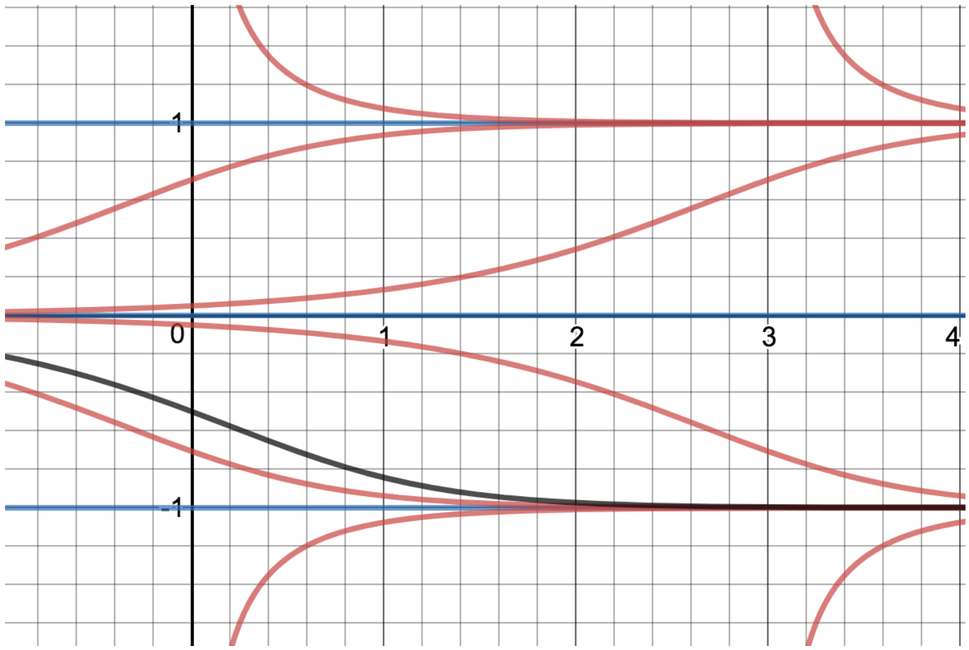
$$\boxed{y = 1 \text{ is a } \textit{stable} \text{ equilibrium}}$$

(b)

The phase line looks like:



and the solutions look like:



Here the equilibria are in blue, and solutions are in red and black.

If the initial condition is $y(0) = -\frac{1}{2}$, then the solution $y(t)$ will approach the nearest stable equilibrium at $y = -1$.
This specific solution $y(t)$ is the one graphed in black above.

Answer to Question 6.
(a) Since the raindrop is spherical, it has surface area $SA = 4\pi r^2$ and volume $V = \frac{4}{3}\pi r^3$.
Since the raindrop is evaporating, it is losing mass M at a rate proportional to its surface area. In terms of differential equations, this means

$$\frac{dM}{dt} = k(SA) = k4\pi r^2$$

for some (negative) constant k.
We can also write its mass in terms of its density ρ as

$$M = \rho V = \rho\left(\frac{4}{3}\pi r^3\right)$$

differentiating both sides and using the chain rule,

$$\frac{dM}{dt} = \frac{d}{dt}\left[\frac{4}{3}\pi r^3 \rho\right] = 4\pi r^2 \rho \frac{dr}{dt}$$

Putting our two equations for $\frac{dM}{dt}$ together,

$$\frac{dM}{dt} = k4\pi r^2 = 4\pi r^2 \rho \frac{dr}{dt}$$

then solving for $\frac{dr}{dt}$,

$$\boxed{\frac{dr}{dt} = \frac{k}{\rho} = c}$$

meaning that the rate at which the radius is decreasing is a constant, where said constant is inversely proportional to the density of water.

(b) Here we want to solve the equation

$$\frac{dv}{dt} + \frac{3c}{(ct + r_0)}v = g$$

since the raindrop starts at rest, the initial condition is

$$v(0) = 0$$

Since our equation is first-order and linear in v, we can solve it using integrating factors.
So we multiply everything by a function $\mu(t)$:

$$\mu\frac{dv}{dt} + \frac{3c\mu}{(ct + r_0)}v = g\mu$$

In order to get the left hand side to look like the result of a product rule, we need μ to satisfy

$$\frac{d\mu}{dt} = \frac{3c\mu}{ct + r_0}$$

this equation is separable:

$$\begin{aligned}\int \frac{d\mu}{\mu} &= \int \frac{3c}{ct + r_0}dt \\ \ln(\mu) &= 3\ln(ct + r_0) \\ \mu(t) &= (ct + r_0)^3\end{aligned}$$

(since we just want *a* integrating factor, we can ignore the constant in the integration above.)
Multiplying everything by our integrating factor μ,

$$\begin{aligned}(ct + r_0)^3\frac{dv}{dt} + 3c(ct + r_0)^2 v &= g(ct + r_0)^3 \\ \left[(ct + r_0)^3 v\right]' &= g(ct + r_0)^3 \\ (ct + r_0)^3 v &= \frac{g}{4c}(ct + r_0)^4 + C_0 \\ v(t) &= \frac{g}{4c}(ct + r_0) + \frac{C_0}{(ct + r_0)^3}\end{aligned}$$

Now we plug in the initial condition $v(0) = 0$ to solve for the constant C_0:

$$v(0) = \frac{g}{4c}(r_0) + \frac{C_0}{r_0^3} = 0$$

$$C_0 = \frac{-gr_0^4}{4c}$$

Plugging that back in, our solution $v(t)$ is given by:

$$\boxed{v(t) = \frac{g}{4c}\left[(ct + r_0) - \frac{r_0^4}{(ct + r_0)^3}\right]}$$

Here are a couple possible ways of explaining why the model is only physically meaningful for $0 \leq t < -r_0/c$:

- The radius r of the raindrop is given by $r = ct + r_0$ (see **(a)**), and after the given time this radius would be negative. This situation isn't physically meaningful; what is the velocity of the raindrop if there is no raindrop?
- Our solution $v(t)$ above is only defined when the $ct + r_0$ term in the denominator is nonzero. At $t = -r_0/c$ this denominator would be zero, meaning $v(t)$ would not be defined at that time. So it wouldn't really make sense to consider $v(t)$ for any t after that point.
- If $ct + r_0$ is negative, then if we look at the original differential equation, it would give us that $\frac{dv}{dt} > g$, i.e. that acceleration is greater than gravity, which cannot be the case.