Math 2930 Worksheet
Exact Equations and Euler's Method

Week 4
February 15, 2019

Learning Goals

- Determine when first-order ODEs are exact.
- Solve first-order exact equations.
- Use integrating factors to make equations exact.
- Analyze the convergence of Euler's method for simple examples.

Questions

Question 1. (a) Find the value(s) of b for which the given equation is exact:

$$(xy^2 + bx^2y) + (x + y)x^2y' = 0$$

(b) Solve it for the value of b you found in part **(a)** .

Question 2. (a) Show that the equation below is *not* exact:

$$x^2y^3 + x(1 + y^2)y' = 0$$

(b) Show that it can be made exact by multiplying both sides of the equation by the integrating factor $\mu(x, y) = \frac{1}{xy^3}$.

(c) Now that the equation is exact, solve it.

Question 3. Consider the equation

$$\frac{dy}{dt} = y$$
$$y(0) = 1$$

(a) Find the analytical solution for $y(t)$ with the given initial condition

(b) If instead we solve the equation using the forward Euler's method, with a step size of h, write down the first 2 iterations. Express your answers in terms of h.

(c) Based on **(b)** , write down the expression after the n-th iteration.

(d) Let n be the number of steps over the interval $[0, t]$, with $n = t/h$, show that in the limit as $h \to 0$, and $n \to \infty$, the numerical answer given by Euler's method converges to the analytical solution that you found in part **(a)** .

Question 4. (a) Show that the equation below is *not* exact:

$$y + (2xy - e^{-2y})y' = 0$$

(b) It turns out that we can make this equation exact by using some sort of integrating factor μ (like we did in question 2). In order to find μ, we'll have to assume that it depends on either x only or on y only, but not both.

Let's assume for now that μ is a function of y only. What differential equation will $\mu(y)$ have to solve in order for our equation to be exact?

(c) What if we tried to look for μ as a function of x only instead. Would this approach work? Why or why not?

(d) Solve the differential equation you found in part **(b)** for $\mu(y)$.

(e) Solve the original equation in part **(a)** using the integrating factor μ you found in part **(d)** .

(f) Can you extend your argument from part **(b)** to more general equations?

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0$$

i.e. what are the conditions on M and N such that an integrating factor $\mu = \mu(x)$ can be used to make the equation exact? When can an integrating factor $\mu = \mu(y)$ be used?

Answer to Question 1.
(a) This equation is exact when:

$$M_y = N_x$$

So for this problem, that becomes:

$$\frac{\partial}{\partial y}\left[xy^2 + bx^2y\right] = \frac{\partial}{\partial x}\left[(x+y)x^2\right]$$

Simplifying and solving for b,

$$2xy + bx^2 = 3x^2 + 2xy$$

$$bx^2 = 3x^2$$

$$\boxed{b = 3}$$

Therefore $b = 3$ is the only value of b for which the given equation is exact.

(b) For $b = 3$, we now want to find a function $\psi(x, y)$ such that:

$$\frac{\partial \psi}{\partial x} = M(x, y) = xy^2 + 3x^2y$$

and

$$\frac{\partial \psi}{\partial y} = N(x, y) = x^3 + x^2y$$

Integrating the first equation with respect to x (while holding y constant), we get:

$$\psi(x, y) = \frac{x^2y^2}{2} + x^3y + f(y)$$

for some function $f(y)$.
Differentiating with respect to y and setting it equal to N, we have

$$\frac{\partial \psi}{\partial y}x^2y + x^3 + f'(y) = N(x, y) = x^3 + x^2y$$

So clearly $f'(y) = 0$, which means that $f(y)$ is a constant.
Therefore the general solution is:

$$\boxed{\psi(x, y) = \frac{x^2y^2}{2} + x^3y = C}$$

Answer to Question 2. (a) For this equation:

$$M(x,y) = x^2y^3, \qquad N(x,y) = x(1+y^2)$$

To check if it's exact, we calculate

$$\frac{\partial M}{\partial y} = 3x^2y^2$$

and

$$\frac{\partial N}{\partial x} = 1 + y^2$$

$$\boxed{M_y \neq N_x, \quad \text{so this equation is not exact.}}$$

(b) Multiplying everything by the integrating factor $\mu(x,y) = \frac{1}{xy^3}$,

$$M(x,y) = x$$

$$\frac{\partial M}{\partial y} = 0$$

and

$$N(x,y) = \frac{1}{y^3} + \frac{1}{y}$$

$$\frac{\partial N}{\partial x} = 0$$

Since $M_y = N_x$, the equation

$$\boxed{x + \left(\frac{1}{y^3} + \frac{1}{y}\right) y' = 0, \quad \text{is exact.}}$$

(in fact it is also separable).

(c) We want to find a function $\psi(x,y)$ with partial derivatives:

$$\frac{\partial \psi}{\partial x} = x, \qquad \frac{\partial \psi}{\partial y} = \frac{1}{y^3} + \frac{1}{y}$$

Integrating the first of those, we get

$$\psi(x,y) = \frac{1}{2}x^2 + f(y)$$

for some function $f(y)$. Integrating the second equation,

$$\psi(x,y) = \frac{-1}{2y^2} + \ln(y) + g(x)$$

Combining these two, we find that the solution is:

$$\boxed{\psi(x,y) = \frac{1}{2}x^2 + \frac{-1}{2y^2} + \ln(y) = C}$$

Answer to Question 3.
(a) We have the equation

$$\frac{dy}{dt} = y$$

This is separable, so we solve it as follows:

$$\int \frac{dy}{y} = \int dt$$
$$\ln(y) = t + C$$
$$y = Ce^t$$

plugging in the initial condition $y(0) = 1$, we get that $C = 1$, resulting in an analytical solution of

$$\boxed{y(t) = e^t}$$

(b) With Euler's method, we start with the initial condition:

$$y_0 = \boxed{1}$$

The first step gives

$$y_1 = y_0 + hf(y_0) = y_0 + hy_0 = (1+h)y_0 = \boxed{1+h}$$

then the second step gives

$$y_2 = y_1 + hf(y_1) = y_1 + h(y_1) = (1+h)y_1 = (1+h)(1+h) = \boxed{(1+h)^2}$$

(c) From **(b)** we see that there is a pattern that

$$y_{n+1} = (1+h)y_n$$

resulting in the formula:

$$\boxed{y_n = (1+h)^n}$$

(d) Taking our answer from **(c)** and replacing n with t/h, we have

$$y_n = (1+h)^{t/h}$$

Then we are interested in taking the limit as $h \to 0$

$$\lim_{h\to 0} y_n = \lim_{h\to 0} (1+h)^{t/h}$$

To figure out this limit, it helps to take the logarithm of both sides,

$$\lim_{h\to 0} \ln(y_n) = \lim_{h\to 0} \ln\left[(1+h)^{t/h}\right] = \lim_{h\to 0} \frac{t\ln(1+h)}{h}$$

this can then be evaluated using L'hôpital's rule,

$$\lim_{h\to 0} \ln(y_n) = \lim_{h\to 0} \frac{t\frac{1}{1+h}}{1} = t$$

Then since

$$\lim_{h\to 0} \ln(y_n) = t$$

we have

$$\boxed{\lim_{h\to 0} y_n = e^t}$$

matching our solution from part **(a)** .
Note: you can also do this problem by taking the limit as $n \to \infty$ instead of $h \to 0$ in a very similar way, but the details are a little trickier.

Answer to Question 4. (a) For this equation,

$$M(x,y) = y, \qquad N(x,y) = 2xy - e^{-2y}$$

To check if it is exact, we compute

$$\frac{\partial M}{\partial y} = 1$$

and

$$\frac{\partial N}{\partial x} = 2y$$

$$\boxed{M_y \neq N_x, \quad \text{so this equation is not exact.}}$$

(b) So we want to come up with an integrating factor μ so that if we multiply the entire equation by μ, then our equation is now exact.

Let's try an integrating factor of the form $\mu = \mu(y)$. Our equation becomes

$$y\mu(y) + \mu(y)\left(2xy - e^{-2y}\right)\frac{dy}{dx} = 0$$

Now we check the condition for exactness:

$$\frac{\partial M}{\partial y} = y\mu'(y) + \mu(y)$$

$$\frac{\partial N}{\partial x} = \mu(y)2y$$

So this equation is exact if:

$$y\mu'(y) + \mu(y) = \mu(y)2y$$

After rearranging,

$$\boxed{\mu'(y) + \left(\frac{1}{y} - 2\right)\mu(y) = 0}$$

Which we see is a first-order linear ODE in y. So if $\mu(y)$ is a solution of this ODE, it is an integrating factor for the original equation.

(c) Now, let's assume that $\mu = \mu(x)$.
Our equation becomes

$$y\mu(x) + \mu(x)\left(2xy - e^{-2y}\right)\frac{dy}{dx} = 0$$

Now we check the condition for exactness:

$$\frac{\partial M}{\partial y} = \mu(x)$$

$$\frac{\partial N}{\partial x} = \mu'(x)\left(2xy - e^{-2y}\right) + \mu(x)2y$$

So this equation is exact if:

$$\mu(x) = \mu'(x)\left(2xy - e^{-2y}\right) + \mu(x)2y$$

Since this expression depends on both x and y, there is not a function $\mu(x)$ that will satisfy this equation.

(d) This linear ODE can be solved as usual, using (another) integrating factor. This means we want to multiply both sides by some function η so that the left hand side looks like a product rule.

$$\eta\mu'(y) + \eta\left(\frac{1}{y} - 2\right)\mu(y) = 0$$

In order for this to be a product rule, we would need

$$\frac{d\eta}{dy} = \eta\left(\frac{1}{y} - 2\right)$$

This can be separated:

$$\int \frac{d\eta}{\eta} = \int \left(\frac{1}{y} - 2\right) dy$$

Solving for η,

$$\ln(\eta) = \ln(y) - 2y + C$$
$$\eta = Cye^{-2y}$$

Since the constant doesn't matter for an integrating factor, we'll just take $C = 1$. So plugging this back in, the ODE for $\mu(y)$ becomes:

$$ye^{-2y}\mu'(y) + \left(e^{-2y} - 2ye^{-2y}\right)\mu(y) = 0$$

The left hand side is now in the form of a product rule:

$$\left(ye^{-2y}\mu(y)\right)' = 0$$

Integrating both sides,

$$ye^{-2y}\mu(y) = C$$

Again, since μ is an integrating factor, the choice of constant C does not matter, so we will take $C = 1$ for simplicity. Then we find the integrating factor is:

$$\boxed{\mu(y) = \frac{e^{2y}}{y}}$$

(e) Multiplying both sides of the original equation by the integrating factor $\mu(y) = \frac{e^{2y}}{y}$,

$$e^{2y} + \left(2xe^{2y} - \frac{1}{y}\right)\frac{dy}{dx} = 0$$

We can check that the equation is now exact:

$$\frac{\partial M}{\partial y} = 2e^{2y}$$

$$\frac{\partial N}{\partial x} = 2e^{2y}$$

So we want to look for a solution of the form $\psi(x, y) = C$. ψ must have partial derivatives:

$$\frac{\partial \psi}{\partial x} = M(x, y) = e^{2y}$$

$$\frac{\partial \psi}{\partial y} = N(x, y) = 2xe^{2y} - \frac{1}{y}$$

Integrating the first equation with respect to x,

$$\psi(x, y) = xe^{2y} + f(y)$$

for some function $f(y)$.
Integrating the second equation with respect to y,

$$\psi(x, y) = xe^{2y} - \ln(y) + g(x)$$

for some function $g(y)$.
Putting these two together, we see that the final solution is

$$\boxed{\psi(x, y) = xe^{2y} - \ln(y) = C}$$

(f) In general, if we multiply by an integrating factor $\mu(x)$, we have

$$\mu(x)M(x, y) + \mu(x)N(x, y)\frac{dy}{dx} = 0$$

This equation is exact if:

$$\frac{\partial}{\partial y}\left[\mu(x)M(x, y)\right] = \frac{\partial}{\partial x}\left[\mu(x)N(x, y)\right]$$

$$\mu(x)M_y = \mu'(x)N + \mu(x)N_x$$

Rearranging,

$$\mu'(x) = \left(\frac{M_y - N_x}{N}\right)\mu(x)$$

In general, the $\frac{M_y - N_x}{N}$ term can depend on both x and y. In order to be able to solve for μ as a function of x, we then need for this term to depend only on x. In other words, an integrating factor of the form $\mu(x)$ can be found if:

$$\boxed{\frac{M_y - N_x}{N} \quad \text{is a function of } x \text{ only}}$$

We can repeat the same process with $\mu(y)$ instead of $\mu(x)$. Our equation becomes:

$$\mu(y)M(x, y) + \mu(y)N(x, y)\frac{dy}{dx} = 0$$

Checking for exactness,

$$\frac{\partial}{\partial y}\left[\mu(y)M(x, y)\right] = \frac{\partial}{\partial x}\left[\mu(y)N(x, y)\right]$$

$$\mu(y)M_y + \mu'(y)M = \mu(y)N_x$$

Rearranging,

$$\mu'(y) = \left(\frac{N_x - M_y}{M} \right) \mu(y)$$

So an integrating factor of the form $\mu(y)$ can be used if and only if:

$$\boxed{\frac{N_x - M_y}{M} \quad \text{is a function of } y \text{ only}}$$