Math 2930 Worksheet
Wave Equation
D'Alembert's Formula

Week 13
April 26th, 2019

D'Alembert's Formula

For the wave equation:

$$a^2 u_{xx} = u_{tt}$$

it turns out that solutions can always be written as:

$$u(x,t) = F(x+at) + G(x-at)$$

for some functions F and G. This worksheet is designed to guide you through the process of using this formula to solve wave equation problems.

Question 1. *D'Alembert's Formula - Initial Displacement*
Consider the wave equation

$$a^2 u_{xx} = u_{tt}$$

(a) Show that $u(x,t) = F(x+at) + G(x-at)$ satisfies the wave equation.

Suppose the initial conditions are:

$$u(x,0) = f(x), \qquad u_t(x,0) = 0$$

in other words, the initial displacement is $f(x)$ and the initial velocity is zero.

(b) Using the fact that the solution $u(x,t)$ can be written in the form $u(x,t) = F(x+at)+G(x-at)$, show that:

$$F(x) + G(x) = f(x)$$
$$aF'(x) - aG'(x) = 0$$

(c) Use the equations from part **(b)** to show that

$$u(x,t) = \frac{f(x+at) + f(x-at)}{2}$$

solves the wave equation with the given initial conditions.

Parts **(a)** through **(c)** assume we have an infinitely long string, i.e. no boundary conditions.

Let h be the function obtained by extending f into $(-L,0)$ as an odd function, and to other values of x as a periodic function of period $2L$. That is,

$$h(x) = \begin{cases} f(x), & 0 \le x \le L, \\ -f(-x), & -L < x < 0; \end{cases}$$
$$h(x+2L) = h(x)$$

(d) Show that:

$$u(x,t) = \frac{h(x-at) + h(x+at)}{2}$$

also satisfies the boundary conditions:

$$u(0,t) = 0, \qquad u(L,t) = 0$$

Question 2. *D'Alembert's Formula versus Separation of Variables*
Consider the wave equation problem

$$a^2 u_{xx} = u_{tt}$$

with boundary conditions

$$u(0,t) = u(L,t) = 0$$

and initial conditions

$$u(x,0) = f(x), \qquad u_t(x,0) = 0$$

(a) Use separation of variables to show that the solution is of the form

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{an\pi t}{L}\right)$$

(b) Use the trigonometric identity

$$\sin(A)\cos(B) = \frac{\sin(A+B) + \sin(A-B)}{2}$$

to write the solution from part **(a)** in the form $F(x+at) + G(x-at)$

(c) Show that your solution from problem **(1d)** is equivalent to your solution from **(2b)**.

[*Hint*: Think about the Fourier series expansion of h]

Question 3. *D'Alembert's Formula - Initial Velocity* Consider the wave equation

$$a^2 u_{xx} = u_{tt}$$

in an infinite one-dimensional medium subject to the initial conditions

$$u(x,0) = 0$$
$$u_t(x,0) = g(x)$$

in other words, the initial displacement is zero and the initial velocity is $g(x)$.

(a) Using the fact that the solution $u(x,t)$ can be written in the form $u(x,t) = F(x+at)+G(x-at)$, show that:

$$F(x) + G(x) = 0$$
$$aF'(x) - aG'(x) = g(x)$$

(b) Use the equations from part **(a)** to show that

$$2aF'(x) = g(x)$$

and therefore that $F(x)$ is given by

$$F(x) = \frac{1}{2a}\int_{x_0}^{x} g(\xi)d\xi + F(x_0)$$

where x_0 is arbitrary.

(c) Show that $G(x)$ is given by:

$$G(x) = -\frac{1}{2a}\int_{x_0}^{x} g(\xi)d\xi - F(x_0)$$

(d) Show that the final solution to this wave equation problem is

$$u(x,t) = \frac{1}{2a}\int_{x-at}^{x+at} g(\xi)d\xi$$

(e) Show that the solution of the problem

$$\begin{aligned} a^2 u_{xx} &= u_{tt} \\ u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \end{aligned}$$

is actually the sum of your answers from questions **(1c)** and **(3d)**:

$$u(x,t) = \frac{f(x-at)+f(x+at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} g(\xi)d\xi$$

Answer to Question 1. (a)
We start with the formula:

$$u(x,t) = F(x+at) + G(x-at)$$

Taking partial derivatives with respect to x, using the Chain Rule,

$$\begin{aligned}\frac{\partial u}{\partial x} &= F'(x+at)\frac{\partial}{\partial x}(x+at) + G'(x-at)\frac{\partial}{\partial t}(x-at)\\ &= F'(x+at) + G'(x-at)\\ \frac{\partial^2 u}{\partial x^2} &= F''(x+at) + G''(x-at)\end{aligned}$$

Similarly, we can take partial derivatives with respect to t using the Chain Rule:

$$\begin{aligned}\frac{\partial u}{\partial t} &= F'(x+at)\frac{\partial}{\partial t}(x+at) + G'(x-at)\frac{\partial}{\partial t}(x-at)\\ &= aF'(x+at) - aG'(x-at)\\ \frac{\partial^2 u}{\partial t^2} &= a^2F''(x+at) + a^2G''(x-at)\end{aligned}$$

So we see that

$$\boxed{a^2\frac{\partial^2 u}{\partial x^2} = a^2F''(x+at) + a^2G''(x-at) = \frac{\partial^2 u}{\partial t^2}}$$

In other words, this means that $u(x,t)$ is a solution of the wave equation.

(b) Starting with

$$u(x,t) = F(x+at) + G(x-at)$$

and plugging in $t=0$,

$$u(x,0) = F(x+0) + G(x-0)$$

and since our initial conditions say that $u(x,0) = f(x)$, we get one equation:

$$\boxed{F(x) + G(x) = f(x)}$$

For the other equation, we also start with

$$u(x,t) = F(x+at) + G(x-at)$$

then taking a partial derivative with respect to t as in **(a)** ,

$$u_t(x,t) = aF'(x+at) - aG'(x-at)$$

Plugging in $t=0$,

$$u_t(x,0) = aF'(x+0) - aG'(x-0)$$

then using the initial condition that $u_t(x,0) = 0$,

$$\boxed{aF'(x) - aG'(x) = 0}$$

(c) From **(b)** we have the system of equations:

$$F(x) + G(x) = f(x)$$
$$aF'(x) - aG'(x) = 0$$

First we rearrange the second equation,

$$aF'(x) = aG'(x)$$
$$F'(x) = G'(x)$$

then integrating both sides,

$$F(x) = G(x) + C$$

for some constant C. Plugging this into our first equation,

$$\begin{aligned} F(x) + G(x) &= f(x) \\ (G(x) + C) + G(x) &= f(x) \\ 2G(x) &= f(x) - C \\ G(x) &= \frac{f(x) - C}{2} \end{aligned}$$

which we can then use to find

$$F(x) = G(x) + C = \frac{f(x) - C}{2} + C = \frac{f(x) + C}{2}$$

So plugging $x + at$ into our equation for F, and $x - at$ into our equation for G,

$$\begin{aligned} u(x,t) &= F(x + at) + G(x - at) \\ &= \frac{f(x + at) + C}{2} + \frac{f(x - at) - C}{2} \end{aligned}$$

$$\boxed{u(x,t) = \frac{f(x + at) + f(x - at)}{2}}$$

(d) We have from the problem description that our solution is:

$$u(x,t) = \frac{h(x + at) + h(x - at)}{2}$$

Plugging in $x = 0$,

$$u(0,t) = \frac{h(at) + h(-at)}{2}$$

and because h was constructed to be an odd function,

$$\boxed{u(0,t) = \frac{h(at) - h(at)}{2} = 0}$$

For the other boundary condition, we plug in $x = L$,

$$u(L,t) = \frac{h(L - at) + h(L + at)}{2}$$

and because h was constructed to be periodic with period $2L$,

$$h(L - at) = h(L - at - 2L) = h(-L - at)$$

and because h was constructed to be an odd function,

$$\boxed{u(L,t) = \frac{h(-L - at) + h(L + at)}{2} = \frac{-h(L + at) + h(L + at)}{2} = 0}$$

Answer to Question 2.
(a) Using separation of variables, we look for solutions of the form $u(x,t) = X(x)T(t)$. Plugging this into the PDE,

$$a^2 u_{xx} = u_{tt}$$
$$a^2 X''T = XT''$$

dividing both sides by a^2XT,

$$\frac{X''}{X} = \frac{T''}{a^2T}$$

Since the left hand side depends only on x and the right hand side depends only on t, both sides must be equal to the same constant, which I will call λ (you could use $-\lambda$ instead if you prefer, it won't make a difference in the end),

$$\frac{X''}{X} = \frac{T''}{a^2T} = \lambda$$

We can rearrange this to give us a system of two ordinary differential equations,

$$X'' - \lambda X = 0$$
$$T'' - a^2\lambda T = 0$$

The boundary and initial conditions on the PDE will then place some boundary and initial conditions on X and T. For the first boundary condition, we get

$$u(0,t) = 0$$
$$X(0)T(t) = 0$$

If $T(t) = 0$ for all values of t, then we would just get the trivial solution. So instead we set

$$X(0) = 0$$

By similar arguments, we get that

$$u(L,t) = 0 \quad \Longrightarrow \quad X(L) = 0$$

and

$$u_t(x,0) = 0 \quad \Longrightarrow \quad T'(0) = 0$$

(The initial condition $u(x,0) = f(x)$ is nonzero, so it won't tell us anything about solutions to the PDE.)
So $X(x)$ must satisfy the eigenvalue problem:

$$X'' - \lambda X = 0$$
$$X(0) = X(L) = 0$$

For the case when $\lambda > 0$, the characteristic equation is $r^2 - \lambda = 0$, so the roots are $r = \pm\sqrt{\lambda}$ and the general solution is:

$$X(x) = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}$$

Plugging in the first boundary condition $X(0) = 0$,

$$X(0) = c_1 + c_2 = 0$$

which means that $c_1 = -c_2$. Plugging in the second boundary condition $X(L) = 0$,

$$X(L) = c_1 e^{\sqrt{\lambda}L} + c_2 e^{-\sqrt{\lambda}L} = 0$$

Using the fact that $c_1 = -c_2$,

$$c_1 \left(e^{\sqrt{\lambda}L} - e^{-\sqrt{\lambda}L} \right) = 0$$

Since $\lambda > 0$, we have that $e^{\sqrt{\lambda}L} > e^{-\sqrt{\lambda}L}$, so the only way this can be zero is if:

$$c_1 = c_2 = 0$$

which just leads to the trivial solution $X(x) = 0$.
For the case when $\lambda = 0$, our ODE for X is just:

$$X'' = 0$$

which we can just integrate twice to get the general solution

$$X(x) = c_1 x + c_2$$

Plugging in the first boundary condition $X(0) = 0$,

$$X(0) = c_2 = 0$$

using this together with the second boundary condition $X(L) = 0$,

$$X(L) = c_1 L = 0 \quad \Longrightarrow \quad c_1 = 0$$

so this just leads to the trivial solution $X(x) = 0$.
For the case when $\lambda < 0$, it will help to define a new variable μ so that $\lambda = -\mu^2$. Then the general solution is

$$X(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$$

Plugging in the first boundary condition $X(0) = 0$,

$$X(0) = c_1 = 0$$

Using this together with the second boundary condition,

$$X(L) = c_2 \sin(\mu L)$$

Setting $c_2 = 0$ would just lead to the trivial solution, so instead

$$\begin{aligned} \sin(\mu L) &= 0 \\ \mu L &= n\pi, \quad n = 1, 2, 3, \dots \end{aligned}$$

So the eigenvalues are

$$\lambda_n = -\left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \dots$$

with corresponding eigenfunctions

$$X_n = \sin\left(\frac{n\pi}{L} x\right)$$

Plugging our eigenvalues λ back into our differential equation for $T(t)$, we get

$$T'' + \left(\frac{an\pi}{L}t\right) T = 0$$

and the general solution is

$$T(t) = c_1 \cos\left(\frac{an\pi}{L}t\right) + c_2 \sin\left(\frac{an\pi}{L}t\right)$$

Since there is no initial velocity, we also have the initial condition $T'(0) = 0$ from before, which means

$$T'(0) = c_2 \frac{L}{an\pi} = 0$$

so we get that $c_2 = 0$. Therefore our solutions $T(t)$ are

$$T_n(t) = c_1 \cos\left(\frac{an\pi}{L}t\right)$$

and our fundamental solutions are

$$u_n(x,t) = X_n(x)T_n(t) = \sin\left(\frac{n\pi}{L}x\right) \cos\left(\frac{an\pi}{L}t\right)$$

And the general solution is a linear combination of the fundamental solutions:

$$u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t)$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L}x\right) \cos\left(\frac{an\pi}{L}t\right)}$$

The initial condition also requires that

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L}x\right) = f(x)$$

on the interval $x \in [0, L]$. So c_n are the coefficients of the Fourier sine series expansion of $f(x)$.

(b) Using the given trig identity,

$$\sin\left(\frac{n\pi}{L}x\right) \cos\left(\frac{an\pi}{L}t\right) = \frac{1}{2} \sin\left(\frac{n\pi}{L}(x+at)\right) + \frac{1}{2} \sin\left(\frac{n\pi}{L}(x-at)\right)$$

$$\sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L}x\right) \cos\left(\frac{an\pi}{L}t\right) = \frac{1}{2} \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L}(x+at)\right) + \frac{1}{2} \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L}(x-at)\right)$$

So we see that this splits up as $F(x+at) + G(x-at)$ where

$$\boxed{F(x) = G(x) = \frac{1}{2} \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L}x\right)}$$

(c) We recall from **(1d)** that the solution to this wave equation is

$$u(x,t) = \frac{h(x+at) + h(x-at)}{2}$$

where $h(x)$ is the odd periodic extension of $f(x)$.
We can also notice from part **(a)** that the coefficients c_n are chosen such that they form the Fourier sine series expansion of $f(x)$. This means that they converge to the exact same odd periodic extension of $f(x)$:

$$h(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L}x\right)$$

It follows that

$$F(x) = G(x) = \frac{h(x)}{2}$$

And our functions F and G from part **(b)** satisfy

$$F(x+at) + G(x-at) = \frac{h(x+at) + h(x-at)}{2}$$

In other words. the two solutions are equivalent.

Answer to Question 3. (a) Again, our solutions are of the form

$$u(x,t) = F(x+at) + G(x-at)$$

Plugging in $t = 0$, we get that

$$u(x,0) = F(x+0) + G(x-0) = 0$$

which gives our first equation

$$\boxed{F(x) + G(x) = 0}$$

For the second equation, we first take the partial derivative with respect to t to get

$$u_t(x,t) = aF'(x+at) - aG'(x-at)$$

then plugging in $t = 0$,

$$u_t(x,0) = aF'(x+0) - aG'(x-0) = g(x)$$

which gives our second equation

$$\boxed{aF'(x) - aG'(x) = g(x)}$$

(b) The first equation gives

$$F(x) = -G(x)$$

which means that after taking the derivative of both sides,

$$F'(x) = -G'(x)$$

Using that in the second equation,

$$aF'(x) - aG'(x) = 2aF'(x) = g(x)$$

Dividing by $2a$,

$$F'(x) = \frac{1}{2a} g(x)$$

Then integrating both sides and using the fundamental theorem of calculus,

$$F(x) - F(x_0) = \int_{x_0}^{x} F'(\xi) d\xi = \frac{1}{2a} \int_{x_0}^{x} g(\xi) d\xi$$

which tells us that $F(x)$ is given by

$$\boxed{F(x) = \frac{1}{2a} \int_{x_0}^{x} g(\xi) d\xi + F(x_0)}$$

(c) Since $F(x) = -G(x)$, we easily get

$$\boxed{G(x) = -\frac{1}{2a} \int_{x_0}^{x} g(\xi) d\xi - F(x_0)}$$

(d) Combining our answers from **(b)** and **(c)** ,

$$\begin{aligned} F(x+at) + G(x-at) &= \frac{1}{2a} \int_{x_0}^{x+at} g(\xi) d\xi + F(x_0) - \frac{1}{2a} \int_{x_0}^{x-at} g(\xi) d\xi - F(x_0) \\ &= \frac{1}{2a} \int_{x-at}^{x_0} + \frac{1}{2a} \int_{x_0}^{x+at} \end{aligned}$$

$$\boxed{u(x,t) = \frac{1}{2a} \int_{x-at}^{x+at} g(\xi) d\xi}$$

(e) There are two different ways of showing this, either using linearity, or just by checking directly. First, the linear way. If we write our two solutions as

$$u(x,t) = \frac{f(x-at) + f(x+at)}{2}, \qquad v(x,t) = \frac{1}{2a} \int_{x-at}^{x+at} g(\xi) d\xi$$

Then since u and v both solve the wave equation, we can check that $u+v$ solves the wave equation

$$a^2(u+v)_{xx} = a^2 u_{xx} + a^2 v_{xx} = u_{tt} + v_{tt} = (u+v)_{tt}$$

and using our answers from the previous questions, we can see that they also solve the initial conditions:

$$u(x,0) + v(x,0) = f(x) + 0 = f(x)$$
$$u_t(x,0) + v_t(x,0) = 0 + g(x) = g(x)$$

We can also check that they solve the boundary conditions directly:

$$u(x,0) = \frac{f(x) + f(x)}{2} + \frac{1}{2a} \int_{x}^{x} g(\xi) d\xi = \frac{2f(x)}{2} + 0 = f(x)$$

and using the fundamental theorem of calculus,

$$
\begin{aligned}
u_t(x,t) &= \frac{-af'(x-at)+af'(x+at)}{2} + \frac{1}{2a}g(x+at)\frac{\partial}{\partial t}(x+at) - \frac{1}{2a}g(x-at)\frac{\partial}{\partial t}(x-at) \\
u_t(x,t) &= \frac{-af'(x-at)+af'(x+at)}{2} + \frac{1}{2}g(x+at) + \frac{1}{2}g(x-at) \\
u_t(x,0) &= \frac{-af'(x)+af'(x)}{2} + \frac{g(x)+g(x)}{2} = g(x)
\end{aligned}
$$

and thus this solves the wave equation and both boundary conditions.