Math 2930 Worksheet
Heat Equation

Week 12
April 19th, 2019

Solving PDEs with Separation of Variables

Given a partial differential equation (PDE) such as the heat equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \qquad u(x,0) = f(x)$$

we can break down the process of solving it into the following steps:

- Use separation of variables, *i.e.* assume that the solutions are a product $u(x,t) = X(x)T(t)$ and reduce the equation to a system of two ordinary differential equations.
- Write down the boundary conditions for $X(x)$ and solve the boundary value problem.
- Find the fundamental solutions and write down the general solution of the PDE.
- Find a formula for the coefficients in the general solution by using the initial condition $u(x,0) = f(x)$.

Question 1. *Heat equation with insulated ends*
Consider a thin pipe placed along the x-axis with ends at $x = 0$ and $x = \pi$. The pipe is filled with a mixture of mostly water, with a small amount of a certain chemical. As the chemical diffuses through the pipe, the concentration $u(x,t)$ of the chemical at location x and time t is governed by the heat equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

Suppose the initial concentration is given by:

$$u(x,0) = x, \qquad x \in [0,\pi].$$

If the ends of the pipe are closed so that none of the chemical can escape, the boundary conditions are:

$$u_x(0,t) = 0, \qquad u_x(\pi,t) = 0, \qquad t \geq 0$$

(a) Assuming that $u(x,t) = X(x)T(t)$, find ordinary differential equations that are satisfied by $X(x)$ and $T(t)$ using separation of variables.

(b) Given the boundary conditions for u:

$$u_x(0,t) = u_x(\pi,t) = 0 \qquad \text{for all } t \geq 0$$

Use these to find boundary conditions for $X(x)$.

(c) Solve the eigenvalue problem for $X(x)$ corresponding to what you found in **(b)** .

(d) For the eigenvalues you found in part **(c)**, solve the corresponding ODE for $T(t)$.

(e) Take a linear combination of all of the fundamental solutions $u_n(x,t)$ to get the general solution $u(x,t)$ of this heat equation.

(f) Now use the initial condition:

$$u(x,0) = x, \qquad x \in [0, L].$$

To find the coefficients in your general solution from part **(e)** .

Question 2. *Heat equation with mixed boundary conditions*

Consider a thin rod of uniform cross-section and homogeneous material placed along the x-axis with ends at $x = 0$ and $x = L$. Heat conduction in the rod is described by the heat equation

$$\alpha^2 u_{xx} = u_t, \qquad x \in [0, L], \qquad t \geq 0$$

Assume one end is kept at constant temperature and one end is insulated. The boundary conditions are

$$u(0, t) = 0, \qquad u_x(L, t) = 0, \qquad t \geq 0$$

Find the general solution of this heat equation problem.

Question 3. *Heat equation in two spatial dimensions*
(a) The heat conduction equation in two spatial dimensions x and y is:

$$\alpha^2(u_{xx} + u_{yy}) = u_t$$

Assuming that $u(x, y, t) = X(x)Y(y)T(t)$, show that $X(x)$, $Y(y)$, and $T(t)$ satisfy the following ordinary differential equations:

$$\begin{aligned} X'' + \mu X &= 0 \\ Y'' + (\lambda - \mu)Y &= 0 \\ T' + \alpha^2 \lambda T &= 0 \end{aligned}$$

where λ and μ are constants.

(b) The heat conduction equation in two space dimensions may also be expressed in polar coordinates as:

$$\alpha^2 \left(u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} \right) = u_t$$

Assuming that $u(r, \theta, t) = R(r)\Theta(\theta)T(t)$, find ordinary differential equations that are satisfied by $R(r)$, $\Theta(\theta)$, and $T(t)$.

Question 4. *Conserved quantities*
Let $u(x,t)$ be a solution of the heat equation $\alpha^2 u_{xx} = u_t$, where $x \in [0, L]$ and $t \geq 0$. We can think of $u(x,t)$ as describing the density of "heat particles" in a metal rod at location x and time t. Then the quantity:

$$E(t) = \int_0^L u(x,t)dx$$

would be the total number of "heat particles" in the rod at time t.

(a) Suppose that the ends of the rod are insulated, *i.e.* the boundary conditions are:

$$u_x(0,t) = 0, \qquad u_x(L,t) = 0$$

Show that $E(t)$ is constant. (Hint: what is $\frac{dE}{dt}$?)

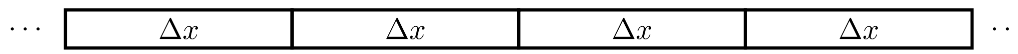
(b) Now suppose that the ends of the rod are connected together, *i.e.* we apply the circular boundary conditions:

$$u(0,t) = u(L,t), \qquad u_x(0,t) = u_x(L,t)$$

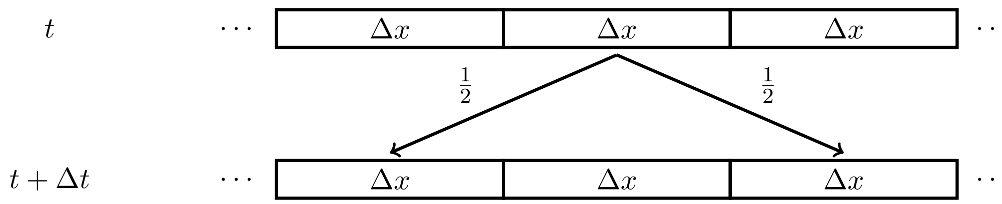
Show that $E(t)$ is constant.

Question 5. *Diffusion as a limit of random walks*

Let the function $v(x,t)$ describe the number of particles at a location x and time t in a one-dimensional settting. Suppose that we divide up our rod into segments of length Δx.



Now, suppose that in an interval of time Δt, each particle moves independently at random a distance Δx to the right with probability $1/2$, and distance Δx to the left with probability $1/2$.



Then $v(x,t)$ satisfies the following equation: [1]

$$v(x,t+\Delta t) = \frac{1}{2}v(x+\Delta x,t) + \frac{1}{2}v(x-\Delta x,t)$$

Assume that $v(x,t)$ is smooth enough that we can expand it in a Taylor series.

Show that if $\Delta x \to 0$ and $\Delta t \to 0$ in such a way that $\frac{(\Delta x)^2}{\Delta t} \to 1$, then $v(x,t)$ satisfies the heat equation:

$$\frac{\partial v}{\partial t} = \frac{1}{2}\frac{\partial^2 v}{\partial t^2}$$

[1] This formula for $v(x,t+\Delta t)$ also describes a method for numerically approximating solutions to the heat equation.

Answer to Question 1. (a)
We start by looking for solutions of the form

$$u(x,t) = X(x)T(t)$$

Plugging this into the heat equation,

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} \\ \frac{\partial}{\partial t}\big[X(x)T(t)\big] &= \frac{\partial^2}{\partial x^2}\big[X(x)T(t)\big] \\ X(x)T'(t) &= X''(x)T(t) \end{aligned}$$

Dividing both sides by $X(x)T(t)$, we get

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)}$$

Since the left hand side depends on t only, and the right hand side depends on x only, both sides must be equal to the same constant, which we will call λ:

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = \lambda$$

Which we can rearrange to get the ODEs:

$$\boxed{\begin{aligned} X''(x) - \lambda X(x) &= 0 \\ T'(t) - \lambda T(t) &= 0 \end{aligned}}$$

(b) The boundary conditions are:

$$u_x(0,t) = u_x(\pi,t) = 0$$

Applying these to $u(x,t) = X(x)T(t)$, we get

$$\begin{aligned} X'(0)T(t) &= 0 \\ X'(\pi)T(t) &= 0 \end{aligned}$$

Having $T(t) = 0$ for all t would just lead to the trivial solution, so instead we have the boundary conditions:

$$\boxed{X'(0) = X'(\pi) = 0}$$

(c) Our eigenvalue problem is:

$$X'' - \lambda X = 0, \qquad X'(0) = X'(\pi) = 0$$

Let's break it down into three cases:

Case 1: $\lambda > 0$ (distinct real roots)

In this case, we have real distinct roots $\pm\sqrt{\lambda}$ of the characteristic equation. The solution is of the form:

$$X(x) = C_1 \cosh(\sqrt{\lambda}x) + C_2 \sinh(\sqrt{\lambda}x)$$

(We could also do this part with $e^{\sqrt{\lambda}x}$ and $e^{-\sqrt{\lambda}x}$ instead of the hyperbolic trig functions). Taking derivatives,

$$X'(x) = \sqrt{\lambda}C_1 \sinh(\sqrt{\lambda}x) + \sqrt{\lambda}C_2 \cosh(\sqrt{\lambda}x)$$

And plugging in the boundary conditions, we get

$$X'(0) = \sqrt{\lambda}C_2 = 0$$

since we assumed that $\lambda > 0$, it follows that $C_2 = 0$. Using this, the other boundary condition is:

$$X'(\pi) = \sqrt{\lambda}C_1 \sinh(\sqrt{\lambda}\pi) = 0$$

Since $\lambda > 0$, and sinh() of something positive is also positive, the only remaining option is

$$C_1 = C_2 = 0$$

so we only get the trivial solution $X(x) = 0$.

Case 2: $\lambda = 0$ (repeated roots)

In this case, our differential equation is just

$$X'' = 0$$

so we can integrate twice to get

$$X(x) = C_1 x + C_2$$

And applying the boundary conditions,

$$X'(0) = X'(\pi) = C_1 = 0$$

so $X = C_2$ is a solution for any constant C_2. We will take $C_2 = 1$, so that we get the followingeigenvalue/eigenfunction pair:

Eigenvalue:	$\lambda_0 = 0$
Eigenfunction:	$X_0 = 1$

Case 3: $\lambda < 0$ (complex roots)

In this case, it will be helpful to define a new variable μ such that

$$\lambda = -\mu^2, \qquad \mu > 0$$

Then the solutions to our differential equation are

$$X(x) = C_1 \cos(\mu x) + C_2 \sin(\mu x)$$

Taking the derivative,

$$X'(x) = -\mu C_1 \sin(\mu x) + \mu C_2 \cos(\mu x)$$

Plugging in the first boundary condition,

$$X'(0) = \mu C_2 = 0$$

since $\mu > 0$ by definition, this means that

$$C_2 = 0$$

Using this when applying the second boundary condition,

$$X'(\pi) = -\mu C_1 \sin(\mu\pi) = 0$$

Since $\mu > 0$, and $C_1 = 0$ would just lead to the trivial solution, we will instead set

$$\begin{aligned} \sin(\mu\pi) &= 0 \\ \mu\pi &= n\pi, \qquad n = 1, 2, 3, \ldots \\ \mu &= n, \qquad n = 1, 2, 3, \ldots \end{aligned}$$

So our eigenvalues and eigenfunctions are:

$$\boxed{\begin{array}{lll} \text{Eigenvalues:} & \lambda_n = -n^2, & n = 1, 2, 3, \ldots \\ \text{Eigenfunctions:} & X_n = \cos(nx) & \end{array}}$$

(d) For $\lambda_0 = 0$, the ODE for T is just:

$$T'(t) = 0$$

The solution to which is clearly

$$\boxed{T_0 = C_0}$$

where C_0 is an arbitrary constant.

For $\lambda_n = -n^2$, the ODE for T is:

$$T' + n^2 T = 0$$

which we can also write as

$$T' = -n^2 T$$

we know that the solution to this is

$$\boxed{T_n(t) = C_n e^{-n^2 t}}$$

where C_n is an arbitrary constant.

(e) Our fundamental solutions are:

$$u_0(x, t) = \frac{C_0}{2}$$

(I added the 1/2 in order to make part **(f)** simpler, but it's not necessary.)
Our other fundamental solutions are:

$$u_n(x, t) = X_n(x)T_n(t) = C_n \cos(nx) e^{-n^2 t}$$

To get the general solution, we take a linear combination of our fundamental solutions:

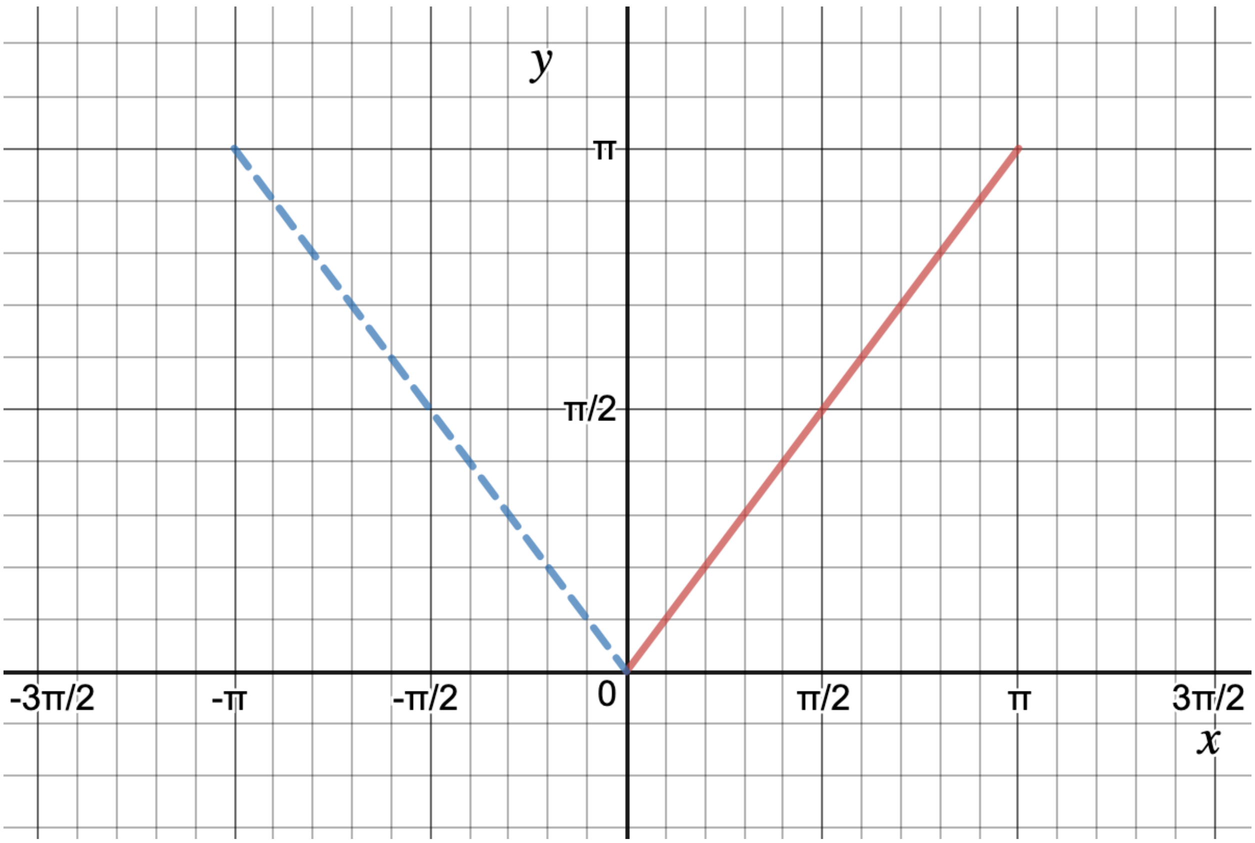
$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t)$$

$$\boxed{u(x,t) = \frac{C_0}{2} + \sum_{n=1}^{\infty} C_n \cos(nx) e^{-n^2 t}}$$

(f) To satisfy the initial condition, we plug in $t = 0$ to get

$$u(x,0) = x = \frac{C_0}{2} + \sum_{n=1}^{\infty} C_n \cos(nx)$$

So we need to find the cosine series expansion of $f(x) = x$.
In order to get the cosine series expansion, we extend $f(x) = x$ so that it is an even function on $[-\pi, \pi]$. The graph would look like:



where the solid red line is the original function, and the blue dashed line is its extension to $[-\pi, 0]$. We can then calculate the cosine series coefficents as the Fourier series coefficients of the above graph:

$$C_0 = \frac{2}{\pi} \int_0^{\pi} x dx = \frac{2}{\pi} \left[\frac{\pi^2}{2} - 0 \right] = \pi$$

and we can calculate the other coefficients as:

$$C_n = \frac{2}{\pi} \int_0^{\pi} x \cos(nx) dx$$

Integrating by parts,

$$U = x, \qquad dV = \cos(nx)dx$$

$$dU = dx, \qquad V = \frac{1}{n} \sin(nx)$$

so

$$
\begin{aligned}
C_n &= \frac{2}{\pi}\left[\frac{x}{n}\sin(nx)\Big|_0^\pi - \int_0^\pi \frac{1}{n}\sin(nx)\right] \\
&= \frac{2}{\pi}\left[\frac{\pi}{n}\sin(n\pi) - \frac{0}{n}\sin(0) + \frac{1}{n^2}\cos(nx)\Big|_0^\pi\right] \\
&= \frac{2}{\pi}\left[\frac{\cos(n\pi) - \cos(0)}{n^2}\right] \\
&= \frac{2}{n^2\pi}\left[(-1)^n - 1\right]
\end{aligned}
$$

Putting these coefficients back into the general solution, our final answer is:

$$
\boxed{u(x,t) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi}\left[(-1)^n - 1\right]\cos(nx)e^{-n^2 t}}
$$

Answer to Question 2. We will begin by using separation of variables. We look for solutions of the form:

$$
u(x,t) = X(x)T(t)
$$

Plugging this into the heat equation,

$$
\begin{aligned}
\alpha^2 u_{xx} &= u_t \\
\alpha^2 (X(x)T(t))_{xx} &= (X(x)T(t))_t \\
\alpha^2 X''(x)T(t) &= X(x)T'(t)
\end{aligned}
$$

Dividing both sides by $\alpha^2 X(x)T(t)$,

$$
\frac{X''(x)}{X(x)} = \frac{T'(t)}{\alpha^2 T(t)}
$$

(You don't have to divide by α^2, but I find it easiest to try and keep the eigenvalue problem for X as simple as possible.) Since the left hand side depends only on x and the right hand side depends only on t, they must both be equal to the same constant λ:

$$
\frac{X''(x)}{X(x)} = \frac{T'(t)}{\alpha^2 T(t)} = \lambda
$$

which we can rearrange to get the following ODEs for $X(x)$ and $T(t)$:

$$
\begin{aligned}
X'' - \lambda X &= 0 \\
T' - \alpha^2 \lambda T &= 0
\end{aligned}
$$

Now we want to turn our boundary conditions for u into boundary conditions for $X(x)$. The first boundary condition is:

$$
\begin{aligned}
u(0,t) &= 0 \\
X(0)T(t) &= 0
\end{aligned}
$$

And since $T(t) = 0$ for all t would just lead to the trivial solution, we will instead set

$$
X(0) = 0
$$

Similarly for the other boundary condition,

$$\begin{aligned} u_x(L,t) &= 0 \\ X'(0)T(t) &= 0 \\ X'(0) &= 0 \end{aligned}$$

So our eigenvalue problem for $X(x)$ is:

$$X'' - \lambda X = 0, \qquad X(0) = X'(L) = 0$$

which we will break down into three cases as usual:

Case 1: $\lambda > 0$ (distinct real roots)

In this case, our general solution is

$$X(x) = C_1 \cosh(\sqrt{\lambda}x) + C_2 \sinh(\sqrt{\lambda}x)$$

Plugging in the first boundary condition,

$$X(0) = C_1 = 0$$

so we have that

$$X(x) = C_2 \sinh(\sqrt{\lambda}x)$$

Taking the derivative,

$$X'(x) = C_2\sqrt{\lambda}\cosh(\sqrt{\lambda}x)$$

Then plugging in the second boundary condition,

$$X'(L) = C_2\sqrt{\lambda}\cosh(\sqrt{\lambda}) = 0$$

Since we assumed that $\lambda > 0$, and cosh() of something positive is also positive, the only possibility is that $C_2 = 0$ and then we just have the trivial solution $X(x) = 0$.

Case 2: $\lambda = 0$ (repeated roots)

In this case, our differential equation is just

$$X'' = 0$$

so we can integrate twice to get the general solution:

$$X(x) = C_1 x + C_2$$

Plugging in the first boundary condition,

$$X(0) = C_2 = 0$$

And plugging in the second boundary condition,

$$X'(L) = C_1 = 0$$

so we only get the trivial solution $X(x) = 0$.

Case 3: $\lambda < 0$ **(complex roots)**

In this case, to make things simpler, I will define a new variable μ such that

$$\lambda = -\mu^2, \qquad \mu > 0$$

Then the general solution is

$$X(x) = C_1 \cos(\mu x) + C_2 \sin(\mu x)$$

Plugging in the first boundary condition,

$$X(0) = C_1 = 0$$

So

$$X(x) = C_2 \sin(\mu x)$$

Taking the derivative,

$$X'(x) = \mu C_2 \cos(\mu x)$$

And plugging in the second boundary condition,

$$X'(L) = \mu C_2 \cos(\mu L) = 0$$

Since $\mu > 0$, and setting $C_2 = 0$ would just lead to the trivial solution, to get something nontrivial we need

$$\begin{aligned} \cos(\mu L) &= 0 \\ \mu L &= (2n-1)\frac{\pi}{2}, \qquad n = 1, 2, 3, \ldots \\ \mu &= \frac{(2n-1)\pi}{2L}, \qquad n = 1, 2, 3, \ldots \end{aligned}$$

So the eigenvalues are:

$$\lambda_n = -\left(\frac{(2n-1)\pi}{2L}\right)^2, \qquad n = 1, 2, 3, \ldots$$

with corresponding eigenfunctions:

$$X_n(x) = \sin\left(\frac{(2n-1)\pi}{2L}x\right)$$

Now we will also solve for $T(t)$ with those eigenvalues λ:

$$T' + \left(\frac{(2n-1)\pi}{2L}\right)^2 T = 0$$

$$T_n(t) = C_n e^{-\left(\frac{(2n-1)\pi}{2L}\right)^2 t}$$

So the fundamental solutions are:

$$u_n(x,t) = X_n(x)T_n(t) = C_n \sin\left(\frac{(2n-1)\pi}{2L}x\right) e^{-\left(\frac{(2n-1)\pi}{2L}\right)^2 t}$$

and the general solution is:

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

$$\boxed{u(x,t) = \sum_{n-1}^{\infty} C_n \sin\left(\frac{(2n-1)\pi}{2L}x\right) e^{-\left(\frac{(2n-1)\pi}{2L}\right)^2 t}}$$

(If you are curious on how you would find those coefficients C_n for a given initial condition, then check out the last problem on last week's worksheet on Fourier series.)

Answer to Question 3. (a) First we assume that $u(x,y,t) = X(x)Y(y)T(t)$. Plugging this into the PDE,

$$\alpha^2 \left(X''YT + XY''T\right) = XYT'$$

Dividng both side by $\alpha^2 XYT$, we get

$$\frac{X''}{X} + \frac{Y''}{Y} = \frac{T'}{\alpha^2 T}$$

Since the left hand side depends only on the x and y while the right hand side depends only on t, both sides must be equal to a constant, which we will call $-\lambda$:

$$\frac{X''}{X} + \frac{Y''}{Y} = \frac{T'}{\alpha^2 T} = -\lambda$$

Which we can use to solve for the ODE for $T(t)$:

$$\boxed{T' + \alpha^2 \lambda T = 0}$$

We also get that

$$\frac{X''}{X} + \frac{Y''}{Y} = -\lambda$$

Rearranging so that all the x terms are on the left hand side, and all the y terms are on the right hand side,

$$\frac{X''}{X} = -\lambda - \frac{Y''}{Y}$$

Both sides must be equal to the same constant, which we will call $-\mu$:

$$\frac{X''}{X} = -\lambda - \frac{Y''}{Y} = -\mu$$

which we can use to get the ODE for $X(x)$:

$$\boxed{X'' + \mu X = 0}$$

and the ODE for $Y(y)$:

$$\boxed{Y'' + (\lambda - \mu)Y = 0}$$

(Note: there are many different acceptable answers here in terms of whether to use λ, $-\lambda$, etc. The ODEs will look slightly different, but the end results will be the same).

(b) Plugging in $u(r,\theta,t) = R(r)\Theta(\theta)T(t)$, and then separating variables

$$\alpha^2 \left(R''\Theta T + \frac{R'\Theta T}{r} + \frac{R\Theta'' T}{r^2}\right) = R\Theta T'$$

$$\frac{R''}{R} + \frac{R'}{rR} + \frac{\Theta''}{r^2\Theta} = \frac{T'}{\alpha^2 T}$$

Since the left hand side depends only on r and θ, while the right hand side depends only on t, both sides must equal the same constant $-\lambda$,

$$\frac{R''}{R} + \frac{R'}{rR} + \frac{\Theta''}{r^2\Theta} = \frac{T'}{\alpha^2 T} = -\lambda$$

This gives us the ODE for $T(t)$:

$$\boxed{T' + \alpha^2\lambda^2 T = 0}$$

Leaving us with

$$\frac{R''}{R} + \frac{R'}{rR} + \frac{\Theta''}{r^2\Theta} = -\lambda$$

Separating the r and θ components,

$$\frac{R''}{R} + \frac{R'}{rR} + \lambda = \frac{-\Theta''}{r^2\Theta}$$
$$\frac{r^2R''}{R} + \frac{rR'}{R} + \lambda r^2 = \frac{-\Theta''}{\Theta} = \mu$$

for some constant μ. This gives us the ODE for $\Theta(\theta)$:

$$\boxed{\Theta'' + \mu\Theta = 0}$$

and the ODE for $R(r)$:

$$\boxed{r^2R'' + rR'' + (\lambda r^2 - \mu)R = 0}$$

(Again, there are many different ways of writing this answer.)

Answer to Question 4. (a) If we take the derivative of E, then we get the derivative (with respect to t) of an integral (with respect to x):

$$\frac{dE}{dt} = \frac{d}{dt}\int_0^L u(x,t)dx$$

Since this derivative and integral are with respect to different variables x and t, we can actually interchange them (a more general form of this is known as Leibniz's formula):

$$\frac{dE}{dt} = \int_0^L \frac{\partial}{\partial t}u(x,t)dx$$

Since u is a solution to the heat equation $u_t = \alpha^2 u_x x$, we can then replace the u_t in our integrand:

$$\frac{dE}{dt} = \alpha^2\int_0^L \frac{\partial^2}{\partial x^2}u(x,t)dx$$

By the fundamental theorem of calculus, we can integrate u_{xx} by just evaluating u_x at the endpoints of the integral:

$$\frac{dE}{dt} = \alpha^2\big[u_x(x,t)\big]_0^L = \alpha^2 u_x(L,t) - \alpha^2 u_x(0,t)$$

If the metal rod is insulated, then both u_x terms are zero:

$$\frac{dE}{dt} = 0 - 0 = 0$$

and therefore $E(t)$ is constant.

(b) By the same argument as part **(a)** ,

$$\frac{dE}{dt} = \alpha^2 u_x(L,t) - \alpha^2 u_x(0,t)$$

and if the ends are connected, both of these terms cancel out, leaving:

$$\frac{dE}{dt} = 0 \qquad \Longrightarrow \qquad E(t) \text{ is constant}$$

Answer to Question 5. If we expand $v(x, t + \Delta t)$ as a Taylor series, we get:

$$v(x, t + \Delta t) = v + \frac{\partial v}{\partial t}\Delta t + \mathcal{O}(\Delta t^2)$$

(in case you haven't seen it before, $\mathcal{O}$ is what's called big-O notation, and it means terms of order Δt^2 or greater, since they will end up disappearing in the limit.)
Similarly, expanding $v(x \pm \Delta x, t)$, we get:

$$\begin{aligned} v(x + \Delta x) &= v + \frac{\partial v}{\partial x}\Delta x + \frac{1}{2}\frac{\partial^2 v}{\partial x^2}(\Delta x)^2 + \mathcal{O}(\Delta x^3) \\ v(x - \Delta x) &= v - \frac{\partial v}{\partial x}\Delta x + \frac{1}{2}\frac{\partial^2 v}{\partial x^2}(\Delta x)^2 + \mathcal{O}(\Delta x^3) \end{aligned}$$

Plugging this all into the formula above and cancelling things out,

$$\begin{aligned} v(x, t + \Delta t) &= \frac{1}{2}v(x + \Delta x, t) + \frac{1}{2}v(x - \Delta x, t) \\ v + \frac{\partial v}{\partial t}\Delta t + \mathcal{O}(\Delta t^2) &= \frac{1}{2}\left[v + \frac{\partial v}{\partial x}\Delta x + \frac{1}{2}\frac{\partial^2 v}{\partial x^2}(\Delta x)^2 + v - \frac{\partial v}{\partial x}\Delta x + \frac{1}{2}\frac{\partial^2 v}{\partial x^2}(\Delta x)^2\right] + \mathcal{O}(\Delta x^3) \\ v + \frac{\partial v}{\partial t}\Delta t + \mathcal{O}(\Delta t^2) &= v + \frac{1}{2}\frac{\partial^2 v}{\partial x^2}(\Delta x)^2 + \mathcal{O}(\Delta x^3) \\ \frac{\partial v}{\partial t}\Delta t + \mathcal{O}(\Delta t^2) &= \frac{1}{2}\frac{\partial^2 v}{\partial x^2}(\Delta x)^2 + \mathcal{O}(\Delta x^3) \\ \frac{\partial v}{\partial t} + \mathcal{O}(\Delta t) &= \frac{1}{2}\frac{\partial^2 v}{\partial x^2}\frac{(\Delta x)^2}{\Delta t} + \mathcal{O}\left(\frac{\Delta x^3}{\Delta t}\right) \end{aligned}$$

Then, taking the limit as $\Delta t \to 0$ and $\frac{(\Delta x)^2}{\Delta t} \to 1$, we are left with:

$$\boxed{\frac{\partial v}{\partial t} = \frac{1}{2}\frac{\partial^2 v}{\partial x^2}}$$

so $v(x, t)$ is a solution to this heat equation.