Math 2930 Worksheet
Fourier Series
Even/Odd Functions

Week 11
April 12th, 2019

Fourier Series Formulas

Given a periodic function $f(x)$ with period $2L$, it may be expanded in a Fourier series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right)$$

where

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx \qquad \text{and} \qquad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

Question 1. Find the Fourier series of the function

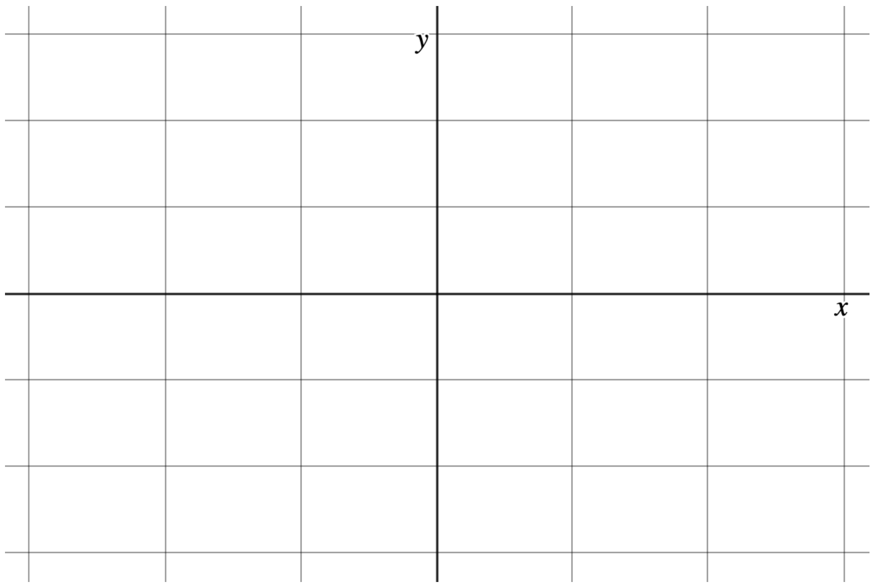
$$f(x) = |x|$$

on the interval $-\pi < x < \pi$.

Question 2. A periodic function is defined by:

$$f(x) = x + \pi, \qquad -\pi \le x < \pi$$
$$f(x + 2\pi) = f(x)$$

(a) Sketch the graph of $f(x)$ for three periods on the axes below. Label important points on the x and y axes.



(b) Find the Fourier series of $f(x)$ on the interval $-\pi < x < \pi$.

Question 3. You are given a continuous function $f(x)$ defined on the interval $[0, L]$. Your goal is to find a Fourier series (satisfying some additional properties) that converges to the given function $f(x)$) on $(0, L)$. The strategy in general is to first extend the function in a clever way and then to compute the Fourier series of that extension.

(a) Suppose that you want to write $f(x)$ as a series of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

i.e. a Fourier series involving only cosine terms. What extension should you use?

(b) Suppose that you want to write $f(x)$ as a series of the form

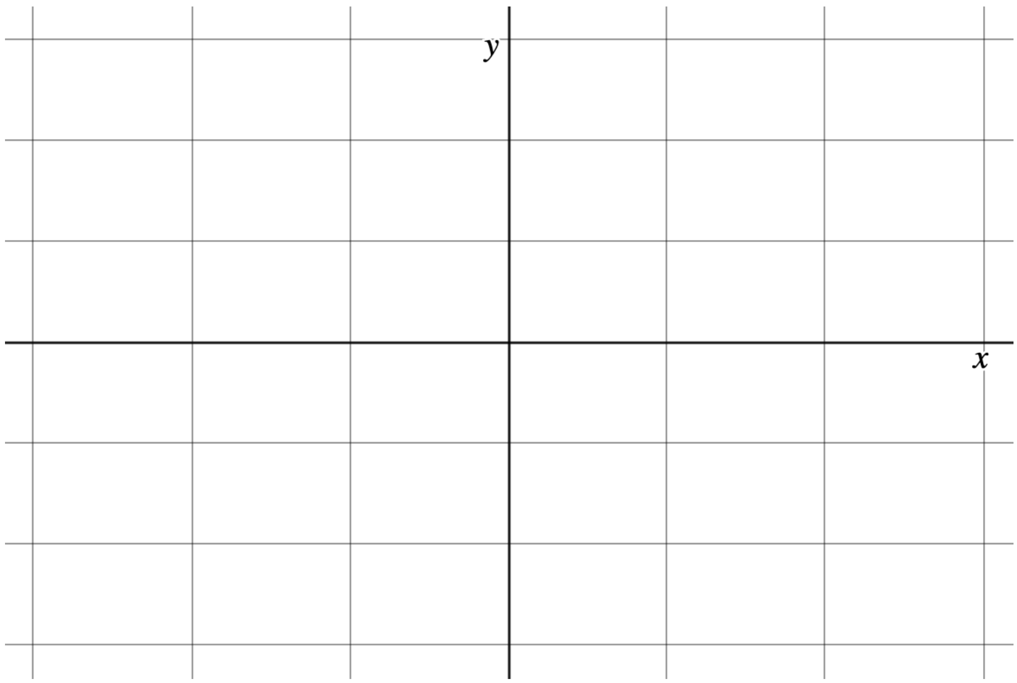
$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

i.e. a Fourier series involving only sine terms. What extension should you use?

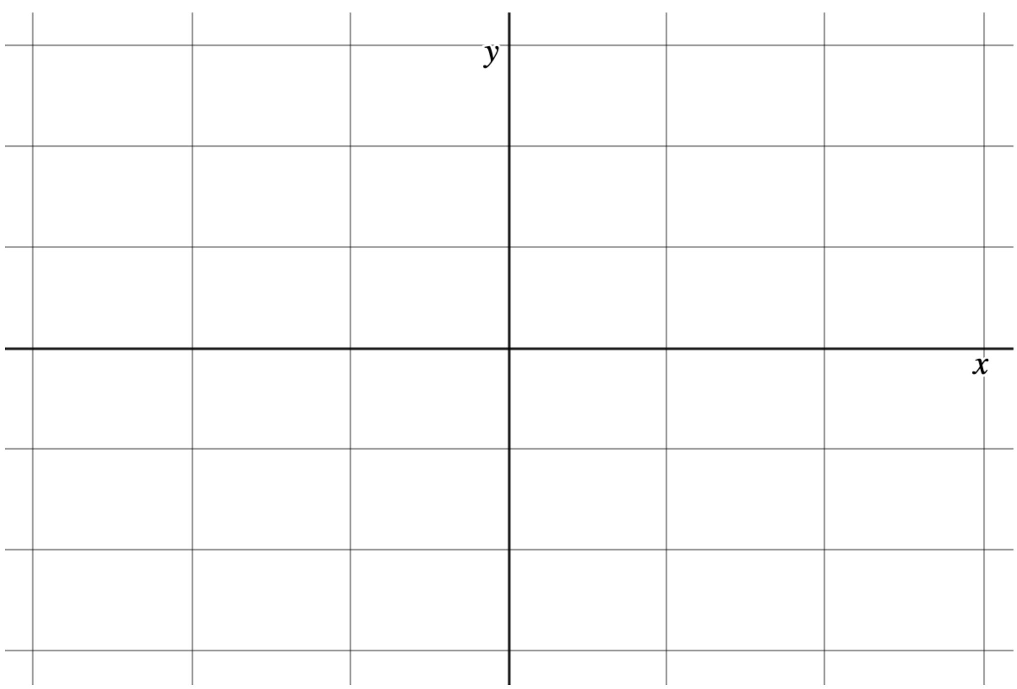
Question 4. A function $f(x)$ is defined on $[0, \pi]$ by

$$f(x) = x, \qquad 0 \leq x \leq \pi/2$$
$$f(x) = 0, \qquad \pi/2 < x \leq \pi.$$

(a) Sketch the even periodic extension of $f(x)$ over $(-3\pi, 3\pi)$ on the axes below. Label important points on the x and y axes.



(b) Sketch the odd periodic extension of $f(x)$ over $(-3\pi, 3\pi)$ on the axes below. Label important points on the x and y axes.



(c) Without doing any calculations, what value does the Fourier cosine series of $f(x)$ converge to at $x = 3\pi/2$?

(d) Without doing any calculations, what value does the Fourier sine series of $f(x)$ converge to at $x = 3\pi/2$?

(e) $f(x)$ can be written as a Fourier sine series,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

Find the coefficients b_n.

Question 5. In solving certain PDE problems using separation of variables, you need to expand a given function $f(x)$ defined on $[0, L]$ as a sum of sine functions with *odd* indices only:

$$\sin\left(\frac{\pi x}{2L}\right), \quad \sin\left(\frac{3\pi x}{2L}\right), \quad \sin\left(\frac{5\pi x}{2L}\right), \quad \dots$$

This question is meant to help walk you through that process.

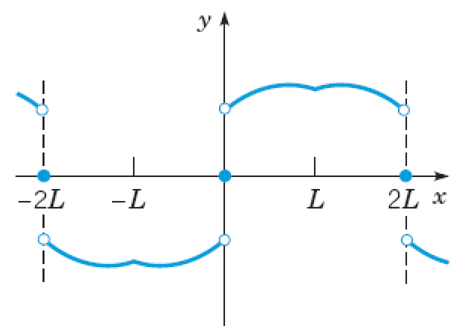
(a) To do this, f should first be extended into $(L, 2L)$ so that it is symmetric about $x = L$. Let the resulting function be extended into $(-2L, 0)$ as an odd function and elsewhere as a periodic function of period $4L$ (see picture below).

Show that this new function has a Fourier series in terms of

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{(2n-1)\pi x}{2L}\right)$$

where

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{(2n-1)\pi x}{2L}\right)dx$$



Answer to Question 1. Since $f(x)$ is defined as going from $-\pi$ to π, we will use $L = \pi$ and $f(x) = |x|$ in our formulas for the Fourier series coefficients.
Also, it will be very useful to note that $f(x) = |x|$ is an even function.
For the constant term, we can calculate a_0 as:

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| dx$$

because $|x|$ is even, we can actually just integrate from 0 to π and then multiply by 2, so

$$a_0 = \frac{2}{\pi} \int_{0}^{\pi} |x| dx$$

And since we are now only integrating over non-negative values of x, we can replace $|x|$ with x, so

$$a_0 = \frac{2}{\pi} \int_{0}^{\pi} x dx$$

$$a_0 = \frac{2}{\pi} \left(\frac{x^2}{2} \right) \Big|_0^{\pi}$$

$$\boxed{a_0 = \pi}$$

For the coefficents b_n of the $\sin(nx)$ terms, they are:

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \sin(nx) dx$$

Since $|x|$ is an even function and sin is an odd function, this is

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (\text{ even function })(\text{ odd function }) dx$$

And since an even function times an odd function is an odd function,

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (\text{ odd function }) dx$$

$$\boxed{b_n = 0}$$

Because the integral of any odd function over $-L$ to L is always zero. This is consistent with the idea that an even function like $|x|$ should only have even terms (such as $\cos(nx)$) in its Fourier Series expansion.
For the coefficients a_n of the $\cos(nx)$ terms, they become:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \cos(nx) dx$$

Because an even function times an even function is even, this is integrating an even function over $-L$ to L so we can just integrate once from 0 to L and then multiply by 2:

$$a_n = \frac{2}{\pi} \int_{0}^{\pi} |x| \cos(nx) dx$$

and since x is only integrated over non-negative values, we can replace $|x|$ with x to get

$$a_n = \frac{2}{\pi}\int_0^\pi x\cos(nx)dx$$

Now we have to integrate by parts:

$$U = x \qquad dV = \cos(nx)dx$$

$$dU = dx \qquad V = \frac{1}{n}\sin(nx)$$

so

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^\pi x\cos(nx)dx = \frac{2}{\pi}\left[\frac{x}{n}\sin(nx)\Big|_0^\pi - \frac{1}{n}\int_0^\pi \sin(nx)dx\right] \\ a_n &= \frac{2}{\pi}\left[0 - 0 + \frac{1}{n^2}(\cos(n\pi) - 1)\right] \\ a_n &= \frac{2}{n^2\pi}(\cos(n\pi) - 1) \end{aligned}$$

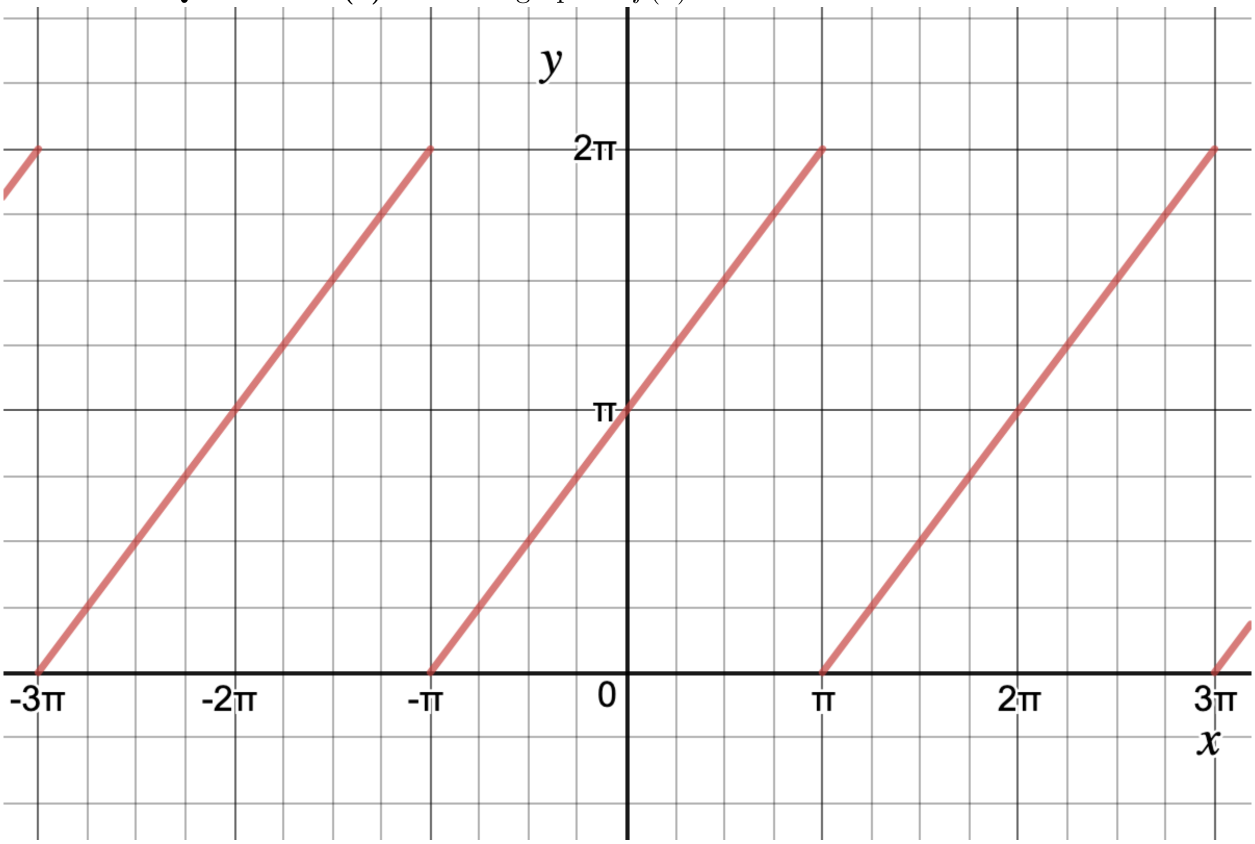
which we can write as

$$\boxed{a_n = \frac{2}{n^2\pi}\left((-1)^n - 1\right)}$$

So putting together all of our coefficients, the Fourier series expansion is:

$$\boxed{|x| = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2}\cos(nx)}$$

Answer to Question 2. (a) Here is a graph of $f(x)$ versus x:



(b) First we calculate a_0 as follows

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+\pi)dx = \frac{1}{\pi}\left[\frac{x^2}{2}+\pi x\right]_{-\pi}^{\pi}$$
$$= \frac{1}{\pi}\left[\frac{\pi^2}{2}+\pi^2-\frac{\pi^2}{2}+\pi^2\right]$$

$$\boxed{a_0 = 2\pi}$$

For the a_n term, we can actually notice that $f(x)$ is just a vertical shift of an odd function, so it will not have any $\cos(nx)$ terms in its Fourier series. So a_n should be zero for all positive n. But we can still compute the coefficients as follows:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+\pi)\cos(nx)dx$$
$$= \frac{1}{\pi}\int_{-\pi}^{\pi}x\cos(nx)dx + \frac{1}{\pi}\int_{-\pi}^{\pi}\cos(nx)dx$$

The first integral is the integral of an odd function from $-\pi$ to π, so it will be zero. So,

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(nx)dx \\ &= \frac{1}{\pi}\cdot\frac{1}{n}\sin(nx)\Big|_{-\pi}^{\pi} \\ &= \frac{1}{n\pi}(\sin(n\pi)-\sin(-n\pi)) \end{aligned}$$

$$\boxed{a_n = 0}$$

Now for the coefficients of the $\sin(nx)$ terms, we get

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}(x+\pi)\sin(nx)dx \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)dx + \int_{-\pi}^{\pi}\sin(nx)dx \end{aligned}$$

Since the first integral is the integral of an even function, this gives

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_{0}^{\pi} x\sin(nx)dx + \frac{-1}{n}\cos(nx)\Big|_{-\pi}^{\pi} \\ &= \frac{2}{\pi}\int_{0}^{\pi} x\sin(nx)dx + \frac{-1}{n}(\cos(n\pi)-\cos(-n\pi)) \\ &= \frac{2}{\pi}\int_{0}^{\pi} x\sin(nx)dx \end{aligned}$$

Integrating by parts,

$$\begin{aligned} U &= x & dV &= \sin(nx) \\ dU &= dx & V &= \frac{-1}{n}\cos(nx) \end{aligned}$$

$$\begin{aligned} b_n &= \frac{2}{\pi}\left[\frac{-x}{n}\cos(nx)\Big|_{0}^{\pi} + \frac{1}{n}\int_{0}^{\pi}\cos(nx)dx\right] \\ &= \frac{2}{\pi}\left[\frac{-\pi}{n}\cos(n\pi) - \frac{1}{n^2}\sin(nx)\Big|_{0}^{\pi}\right] \end{aligned}$$

$$\boxed{b_n = \frac{-2}{n}(-1)^n = \frac{2(-1)^{n+1}}{n}}$$

So putting this all together,

$$\boxed{f(x) = \pi + 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nx)}$$

Answer to Question 3. (a)
For this problem, we want to extend the function f, that is create a new function (which I'll call $g(x)$) which is defined on the whole interval $[-L, L]$.

Since it is an extension, we need that $f(x) = g(x)$ on $[0, L]$, but we will be free to choose what $g(x)$ does from $-L$ to 0.

In order to only get $\cos(n\pi x/L)$ terms when calculating the Fourier series of $g(x)$, we will want to make sure that $g(x)$ is an even function.

This can be done by defining $g(x)$ as:

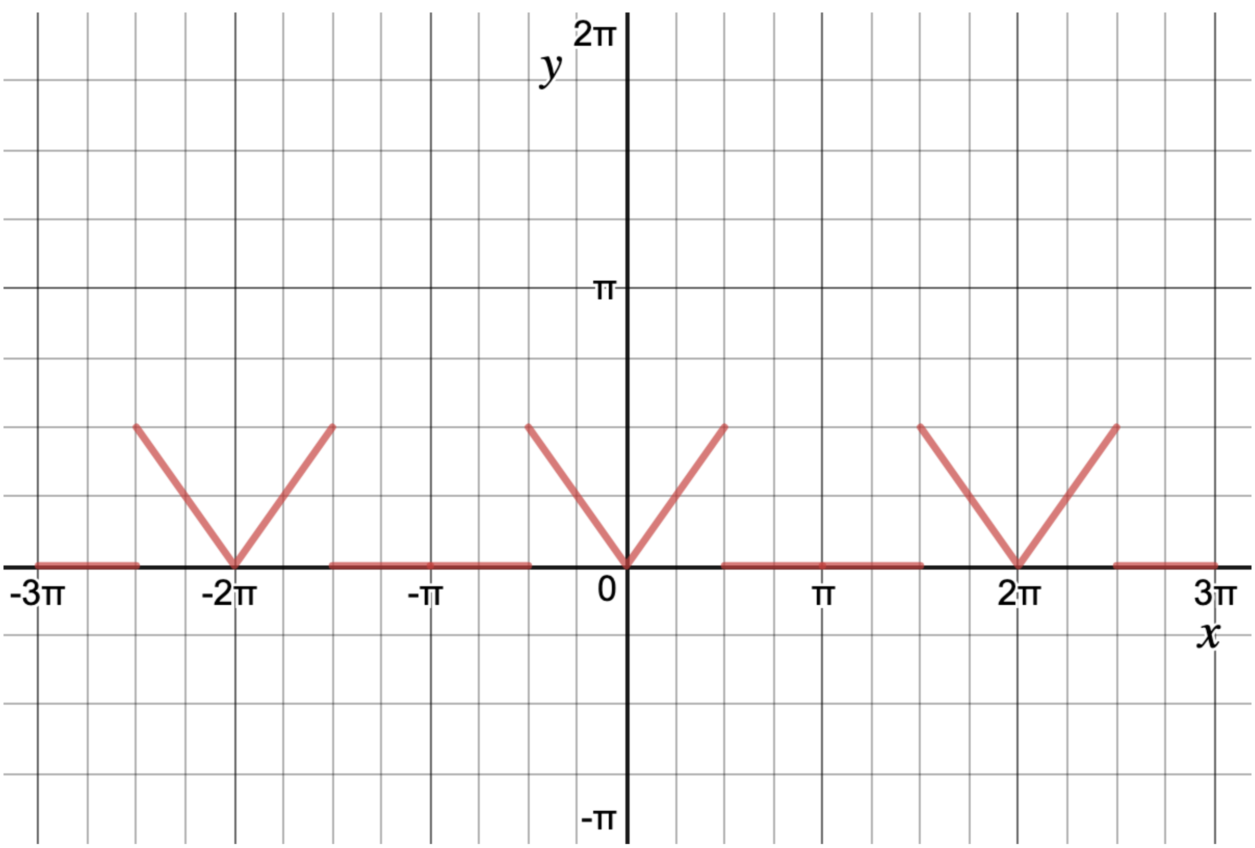
$$\boxed{g(x) = \begin{cases} f(x), & 0 \leq x \leq L \\ f(-x), & -L \leq x < 0 \end{cases}}$$

(b) As in part **(a)**, we want to define a new function $g(x)$ on $[-L, L]$ that is the same as $f(x)$ on $[0, L]$.

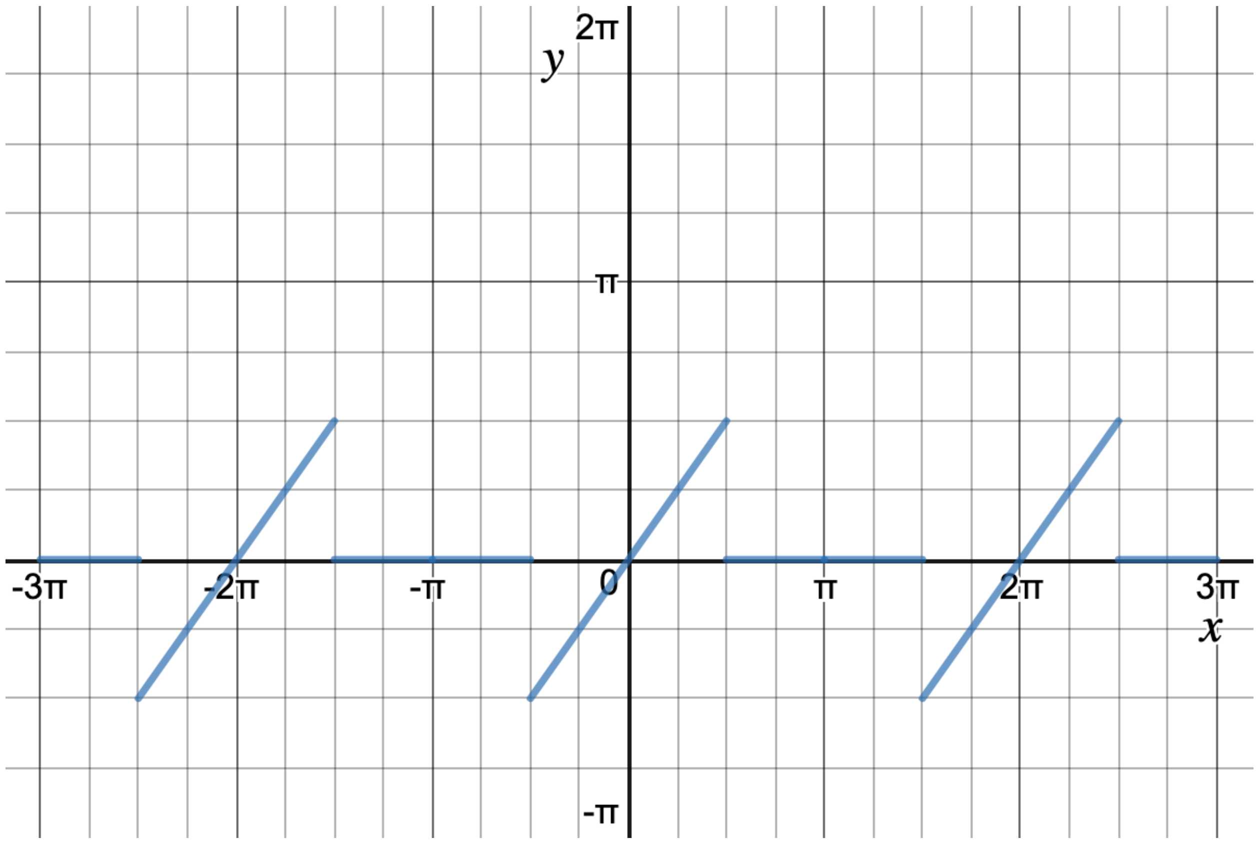
Since we want only $\sin(nx)$ terms in our Fourier series, we will need $g(x)$ to be an odd function. So the extension is:

$$\boxed{g(x) = \begin{cases} f(x), & 0 \leq x \leq L \\ -f(-x), & -L \leq x < 0 \end{cases}}$$

Answer to Question 4. (a) The even periodic extension of f should look like:



(b) The odd periodic extension of f should look like:



(c) We know that our Fourier cosine series is based on the even periodic extension we found in part **(a)** .
If we look back at the graph, we'll see that the function is discontinuous there, jumping from a value of 0 to a value of $\pi/2$. Since the function is discontinuous, the Fourier series will converge to the average of the two points:

$$\frac{1}{2}\left(0+\frac{\pi}{2}\right)=\boxed{\pi/4}$$

(d) We know that our Fourier cosine series is based on the odd periodic extension we found in part **(b)** .
If we look back at the graph, we'll see that the function is discontinuous there, jumping from a value of 0 to a value of $-\pi/2$. Since the function is discontinuous, the Fourier series will converge to the average of the two points:

$$\frac{1}{2}\left(0+\frac{-\pi}{2}\right)=\boxed{-\pi/4}$$

(e) To make sure our Fourier series only has sine terms, we'll want to calculate the Fourier series based on our odd extension from part **(b)**, which we can also write as:

$$f(x) = \begin{cases} 0, & -\pi < x < \frac{-\pi}{2} \\ x, & \frac{-\pi}{2} < x < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < x < \pi \end{cases}$$

Since this function was constructed to be odd, we only need to calculate the coefficients b_n of the sine terms:

$$b_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} x\sin(nx)dx$$

Since $x\sin(nx)$ is an even function, we can instead write this as

$$b_n = \frac{2}{\pi}\int_{0}^{\pi/2} x\sin(nx)dx$$

Integrating by parts,

$$U = x \qquad dV = \sin(nx)$$

$$dU = dx \qquad V = \frac{-1}{n}\cos(nx)$$

$$\begin{aligned} b_n &= \frac{2}{\pi}\left[\frac{-x}{n}\cos(nx)\bigg|_0^{\pi/2} + \frac{1}{n}\int_0^{\pi/2}\cos(nx)dx\right] \\ &= \frac{2}{\pi}\left[\frac{-\pi}{2n}\cos\left(\frac{n\pi}{2}\right) + \frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\right] \end{aligned}$$

$$\boxed{b_n = \frac{-1}{n}\cos\left(\frac{n\pi}{2}\right) + \frac{2}{n^2\pi}\sin\left(\frac{n\pi}{2}\right)}$$

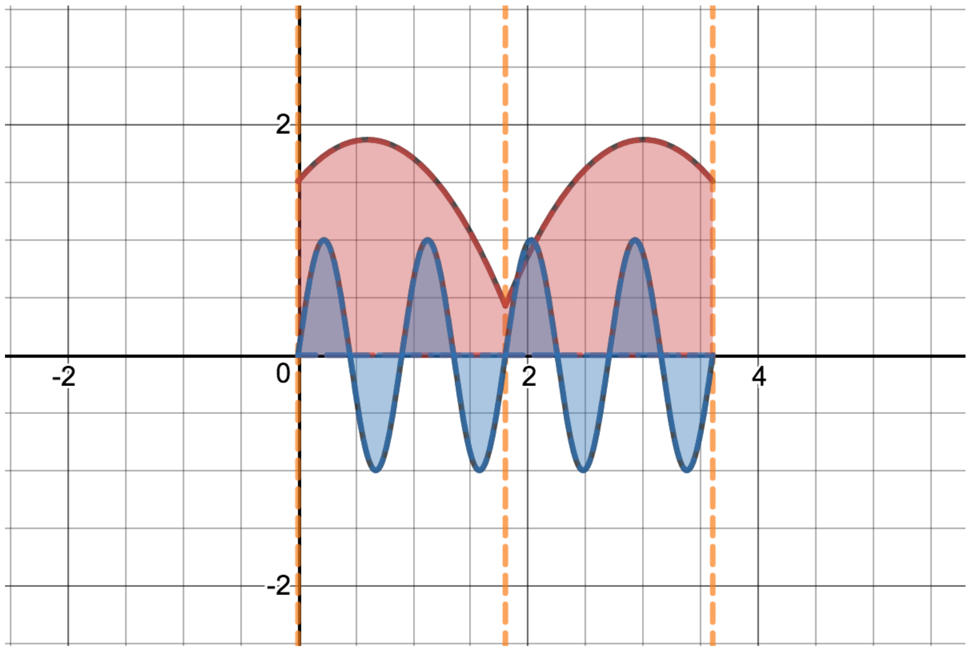
Answer to Question 5. (a) Since the given extension of the function is even, we know that $a_n = 0$ for every n, and the function can be written as a sum of sine functions only.
Calculating the coefficients of these sine terms (using the fact that f is odd):

$$\begin{aligned} b_n &= \frac{1}{2L}\int_{-2L}^{2L} f(x)\sin\left(\frac{n\pi x}{2L}\right)dx \\ &= \frac{2}{2L}\int_{0}^{2L} f(x)\sin\left(\frac{n\pi x}{2L}\right)dx \\ &= \frac{1}{L}\int_{0}^{2L} f(x)\sin\left(\frac{n\pi x}{2L}\right)dx \end{aligned}$$

We can take advantage of the additional symmetry in this problem by breaking down this integral into two pieces, one from 0 to L and the other from L to $2L$ as follows:

$$b_n = \frac{1}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{2L}\right)dx + \frac{1}{L}\int_{L}^{2L} f(x)\sin\left(\frac{n\pi x}{2L}\right)dx$$

Now, for even values of n, the way we have extended f so that it is symmetric about L makes sure that these two integrals cancels out. (See the picture below)

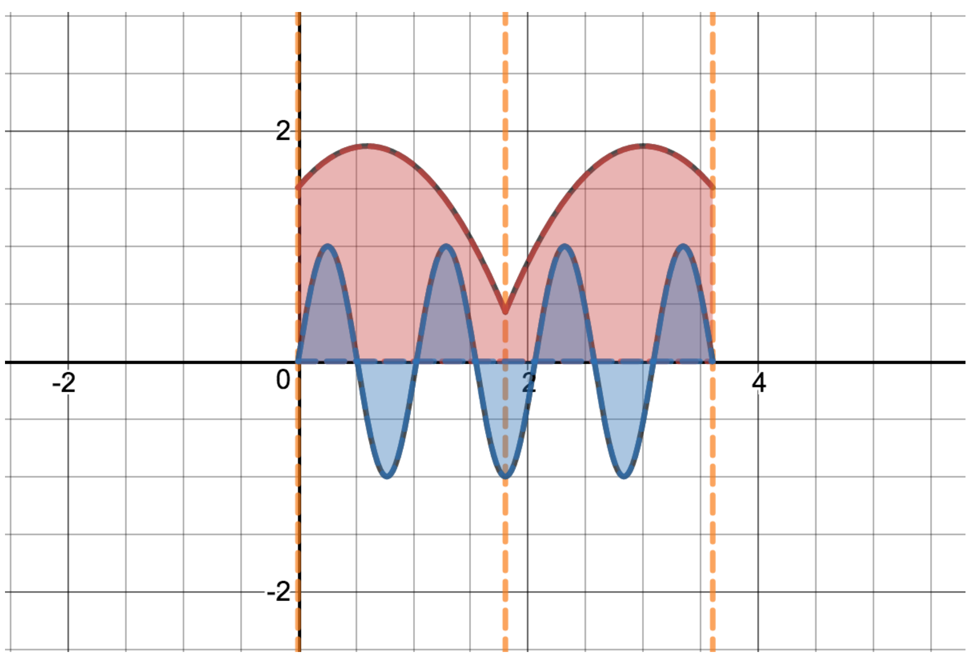


The (even-indexed) sine function is in blue and the extension of f is in red. Looking at this picture, you can see that sin is odd about $x = L$, and f is even about $x = L$, so b_n will be the integral from 0 to $2L$ of a function that is odd about $x = L$.
In other words, the integral from 0 to L and the integral from L to $2L$ above will cancel each other out:

$$b_n = \frac{1}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{2L}\right)dx + \frac{1}{L}\int_L^{2L} f(x)\sin\left(\frac{n\pi x}{2L}\right)dx = 0$$

Thus $b_n = 0$ when n is even.

Now, if n is odd, the picture looks something like:



Now we see that everything is perfectly symmetric about $x = L$, so these two integrals will equal the same value:

$$\begin{aligned} b_n &= \frac{1}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{2L}\right)dx + \frac{1}{L}\int_L^{2L} f(x)\sin\left(\frac{n\pi x}{2L}\right)dx \\ &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{2L}\right)dx \end{aligned}$$

So the b_n coefficients are given by:

$$b_n = \begin{cases} 0, & n \text{ even} \\ \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{2L}\right)dx, & n \text{ odd} \end{cases}$$

Putting this all together, the Fourier series expansion of $f(x)$ is:

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2L}\right) \\ &= \sum_{n \text{ odd}} \left[\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{2L}\right)dx\right]\sin\left(\frac{n\pi x}{2L}\right) \\ &= \sum_{n=1}^{\infty} \left[\frac{2}{L}\int_0^L f(x)\sin\left(\frac{(2n-1)\pi x}{2L}\right)dx\right]\sin\left(\frac{(2n-1)\pi x}{2L}\right) \end{aligned}$$

so it only has the $\sin\left(\frac{(2n-1)\pi x}{2L}\right)$ terms as desired.