Math 2930 Worksheet

Euler Equations
Boundary Value Problems

Week 10
March 29th, 2019

Question 1. Find the eigenvalues and eigenfunctions of:

$$y'' + \lambda y = 0, \qquad y'(0) = y'(\pi) = 0$$

Question 2. Find the solution of:

$$y'' + 3y = \cos(x), \qquad y'(0) = y'(\pi) = 0$$

Question 3. Find the general solution of:

$$x^2y'' + 6xy' + 6y = 0$$

Question 4. Find the eigenvalues and eigenfunctions of:

$$y'' + 4y' + 4\lambda y = 0, \qquad y(0) = y(1) = 0$$

Question 5. Find the general solution of:

$$x^2 \frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = x^5$$

Question 6. (a) Find the general solution of:

$$x^2 y'' + xy' + \lambda y = 0$$

You may assume that $\lambda > 0$.

(b) Find the eigenvalues and eigenfunctions when $y(1) = y(2) = 0$.

Question 7. Find all values of β for which all solutions of

$$x^2y'' + \beta y = 0$$

approach zero as $x \to 0$.

Answer to Question 1. Given the equation $y'' + \lambda y = 0$, we first try to solve for the roots using the characteristic polynomial:

$$\begin{aligned} y'' + \lambda y &= 0 \\ r^2 + \lambda &= 0 \\ r &= \pm\sqrt{-\lambda} \end{aligned}$$

And we notice that we get very different behavior depending upon the sign of λ. To be thorough, we have to check the three following cases.

Case 1: $\lambda < 0$

To make the ensuing algebra easier, let's define a new variable ω so that

$$\lambda = -\omega^2, \quad \omega > 0$$

Then the roots of the characteristic polynomial are:

$$r = \pm\sqrt{-\lambda} = \pm\sqrt{-(-\omega^2)} = \pm\,\omega$$

The corresponding general solution is:

$$y(x) = C_1 e^{\omega x} + C_2 e^{-\omega x}$$

Now we want to plug in the boundary conditions. First, we take the derivative

$$y'(x) = \omega C_1 e^{\omega x} - \omega C_2 e^{-\omega x}$$

Plugging in the first boundary condition of $y'(0) = 0$, we get that

$$\begin{aligned} y'(0) = \omega C_1 - \omega C_2 &= 0 \\ C_1 &= C_2 \end{aligned}$$

(remember that we defined $\omega > 0$ earlier)
Plugging in the second boundary condition $y'(\pi) = 0$, we get that

$$y'(\pi) = \omega C_1 e^{\omega\pi} - \omega C_2 e^{-\omega\pi} = 0$$

Using that $C_1 = C_2$ from the other equation,

$$\omega C_1 \left(e^{\omega\pi} - e^{-\omega\pi}\right) = 0$$

Since $\omega > 0$, it follows that $e^{\omega\pi} > e^{-\omega\pi}$, and so the only way this can hold is if $C_1 = 0$.
But in this case, we have that $C_1 = C_2 = 0$, which is just the trivial solution $y = 0$. While $y = 0$ is a solution of the BVP, it's not a *useful* one.

Since there are no non-trivial solutions for $\lambda < 0$, we would say that there are no eigenvalues for $\lambda < 0$.

Case 2: $\lambda = 0$

In this case, we have repeated roots:

$$r = \pm\sqrt{-\lambda} = 0,\ 0$$

So the corresponding general solution is

$$y(x) = C_1 + C_2 x$$

We of course get that $y' = C_2$, and so our two boundary conditions tell us that:

$$y'(0) = C_2 = 0$$
$$y'(\pi) = C_2 = 0$$

but place no restrictions on C_1.
Therefore $\lambda = 0$ is an eigenvalue, with corresponding solution $y = C$, where C is any constant. Since eigenfunctions are only defined up to a constant, we will not bother writing the C and say that we have:

$$\boxed{\begin{aligned} \text{Eigenvalue: } \lambda_0 &= 0 \\ \text{Eigenfunction: } y_0 &= 1 \end{aligned}}$$

Case 3: $\lambda > 0$

For this case, we will again define ω so that

$$\lambda = \omega^2, \quad \omega > 0$$

Now the roots to our characteristic equation are:

$$r = \pm\sqrt{-\lambda} = \pm\sqrt{-\omega^2} = \pm\omega i$$

and the corresponding general solution is:

$$y(x) = C_1 \cos(\omega x) + C_2 \sin(\omega x)$$

Taking the derivative,

$$y'(x) = -\omega C_1 \sin(\omega x) + \omega C_2 \cos(\omega x)$$

Plugging in the first boundary condition $y'(0) = 0$, we get

$$y'(0) = \omega C_2 = 0$$

And since we defined ω to be nonzero, we get that $C_2 = 0$.
Plugging in the second boundary condition $y'(\pi) = 0$, and using that $C_2 = 0$, we get

$$-\omega C_1 \sin(\omega\pi) = 0$$

We defined ω to be nonzero, and $C_1 = 0$ would lead to the trivial solution $y = 0$. So the only way to get a nontrivial solution is if:

$$\begin{aligned} \sin(\omega\pi) &= 0 \\ \omega\pi &= n\pi, \qquad n = 1, 2, 3, \ldots \\ \omega &= n, \qquad n = 1, 2, 3, \ldots \\ \lambda &= n^2, \qquad n = 1, 2, 3, \ldots \end{aligned}$$

Therefore we have the following eigenvalues and eigenfunctions:

$$\boxed{\begin{array}{c}\text{Eigenvalues: } \lambda_n = n^2 \\ \text{Eigenfunctions: } y_n = \cos(nx) \\ \text{for } \ n = 1, 2, 3, \ldots\end{array}}$$

Answer to Question 2.
This is a non-homogeneous problem, so first let's find the complementary solution y_c by using the roots of the characteristic polynomial:

$$\begin{aligned} y'' + 3y &= 0 \\ r^2 + 3 &= 0 \\ r^2 &= -3 \\ r &= \pm\sqrt{3}i \end{aligned}$$

It follows that the complementary solution is:

$$y_c(x) = C_1 \cos(\sqrt{3}x) + C_2 \sin(\sqrt{3}x)$$

For the particular solution Y, we could use variation of parameters, but I'm going to use the method of undetermined coefficients and guess a particular solution of the form:

$$Y = A\cos(x)$$

(*Note*: In general, we would also want to include a $B\sin(x)$ term whenever we have an $A\cos(x)$ term. For this problem, however, since we only have even derivatives we actually won't need the $\sin(x)$ term. But with undetermined coefficients, it never hurts to include more terms than necessary, it just might make keeping track of all of the terms more tedious.)

Taking derivatives,

$$\begin{aligned} Y' &= -A\sin(x) \\ Y'' &= -A\cos(x) \end{aligned}$$

Plugging this into the original equation,

$$\begin{aligned} Y'' + 3Y &= \cos(x) \\ -A\cos(x) + 3(A\cos(x)) &= \cos(x) \\ 2A\cos(x) &= \cos(x) \\ 2A &= 1 \\ A &= \frac{1}{2} \end{aligned}$$

So the particular solution Y is

$$Y(x) = \frac{1}{2}\cos(x)$$

and the general solution y is

$$y(x) = y_c + Y = C_1 \cos(\sqrt{3}x) + C_2 \sin(\sqrt{3}x) + \frac{1}{2}\cos(x)$$

Now that we have the general solution, we can start plugging in the boundary conditions.
Taking derivatives,

$$y'(x) = -\sqrt{3}C_1 \sin(\sqrt{3}x) + \sqrt{3}C_2 \cos(\sqrt{3}x) - \frac{1}{2}\sin(x)$$

Plugging in the first boundary condition of $y'(0) = 0$, we get that

$$\begin{aligned} y'(0) = 0 + \sqrt{3}C_2 - 0 &= 0 \\ C_2 &= 0 \end{aligned}$$

Plugging in the second boundary condition $y'(\pi) = 0$ (and using that $C_2 = 0$),

$$\begin{aligned} y'(\pi) = -\sqrt{3}C_1 \sin(\sqrt{3}\pi) - \frac{1}{2}\sin(\pi) &= 0 \\ -\sqrt{3}C_1 \sin(\sqrt{3}\pi) &= 0 \end{aligned}$$

And since $\sin(\sqrt{3}\pi) \neq 0$, the only way this can be true is if

$$C_1 = 0$$

So the solution to this boundary value problem is:

$$\boxed{y = \frac{1}{2}\cos(x)}$$

Answer to Question 3.
Since this is an Euler equation, instead of looking for solutions of the form $y = e^{rx}$, we're going to look for solutions of the form $y = x^r$.
Taking derivatives,

$$\begin{aligned} y &= x^r \\ y' &= rx^{r-1} \\ y'' &= r(r-1)x^{r-2} \end{aligned}$$

Plugging this back into the equation,

$$\begin{aligned} x^2y'' + 6xy' + 6y &= 0 \\ x^2\left(r(r-1)x^{r-2}\right) + 6x\left(rx^{r-1}\right) + 6\left(x^r\right) &= 0 \\ \left(r^2 + 5r + 6\right)x^r &= 0 \end{aligned}$$

Since we want this to hold for *all* values of x, our characteristic polynomial will be:

$$r^2 + 5r + 6 = 0$$

which we can factor as:

$$\begin{aligned} (r+3)(r+2) &= 0 \\ r &= -3, -2 \end{aligned}$$

So the general solution is:

$$y = C_1 x^{-2} + C_2 x^{-3} =$$

or alternatively,

$$\boxed{y = \frac{C_1}{x^2} + \frac{C_2}{x^3}}$$

Answer to Question 4.
First, looking for the roots of the characteristic polynomial,

$$y'' + 4y' + 4\lambda y = 0$$
$$r^2 + 4r + 4\lambda = 0$$

Using the quadratic formula,

$$r = \frac{-4 \pm \sqrt{16 - 16\lambda}}{2}$$

which simplifies to:

$$r = -2 \pm 2\sqrt{1 - \lambda}$$

Again, we will have to consider three different cases depending upon the sign of the term inside the square root.

$\boxed{\text{Case 1: } \lambda < 1}$
Let $\lambda = 1 - \omega^2, \quad \omega > 0$ Then we can write the roots as:

$$r = -2 \pm \sqrt{1 - (1 - \omega^2)} = -2 \pm \omega$$

and the corresponding general solution is

$$y(x) = C_1 e^{(-2+2\omega)x} + C_2 e^{(-2-2\omega)x}$$

Now, plugging in the first boundary condition $y(0) = 0$,

$$y(0) = C_1 + C_2 = 0$$
$$C_2 = -C_1$$

Which we can use with the second boundary condition $y(1) = 0$ to get

$$C_1 e^{-2+2\omega} - C_1 e^{-2-2\omega} = 0$$
$$C_1 \left(e^{2\omega} - e^{-2\omega}\right) = 0$$

Since we have defined $\omega > 0$, it follows that $e^{2\omega} > e^{-2\omega}$, so the term inside the parentheses is never zero. Thus the only way of matching the boundary conditions is to have $C_1 = C_2 = 0$, which just gives the trivial solution $y = 0$.

So there are no eigenvalues with $\lambda < 1$.

$\boxed{\text{Case 2: } \lambda = 1}$
In this case, we have repeated roots, since

$$r = -2 \pm \sqrt{1 - \lambda} = -2 \pm 0 = -2, \ -2$$

So the general solution is

$$y = C_1 e^{-2x} + C_2 x e^{-2x}$$

Plugging in the boundary conditions,

$$\begin{aligned} y(0) &= C_1 = 0 \\ y(1) &= C_1 e^{-2} + C_2 e^{-2} = 0 \end{aligned}$$

which leads to $C_1 = C_2 = 0$, which gives the trivial solution.

Case 3: $\lambda > 1$

Now let $\lambda = 1 + \omega^2$, with $\omega > 0$.
Then the roots of our characteristic equation become:

$$\begin{aligned} r &= -2 \pm \sqrt{1-\lambda} = -2 \pm \sqrt{1-(1+\omega^2)} \\ r &= -2 \pm\ 2\omega i \end{aligned}$$

So the general solution is

$$y = C_1 e^{-2x} \cos(2\omega x) + C_2 e^{-2x} \sin(2\omega x)$$

Plugging in the first boundary condition $y(0) = 0$, we get

$$y(0) = C_1 = 0$$

Using this together with the second boundary condition $y(1) = 0$, we get

$$y(1) = C_2 e^{-2} \sin(2\omega) = 0$$

If $C_2 = 0$, then we just get the trivial solution $y = 0$.
So the only way of getting nontrivial solutions is if:

$$\begin{aligned} \sin(2\omega) &= 0 \\ 2\omega &= n\pi, \qquad n = 1, 2, 3, \ldots \\ \omega &= \frac{n\pi}{2}, \qquad n = 1, 2, 3, \ldots \end{aligned}$$

Therefore we have the following eigenvalues and eigenfunctions:

$$\boxed{\begin{gathered} \text{Eigenvalues: } \lambda_n = 1 + \left(\frac{n\pi}{2}\right)^2 \\ \text{Eigenfunctions: } y_n = e^{-2x} \sin(n\pi x) \\ \text{for } \ n = 1, 2, 3, \ldots \end{gathered}}$$

Answer to Question 5.
First, let's look for complementary solutions with the guess $y = x^r$. Taking derivatives,

$$\begin{aligned} y &= x^r \\ y' &= r x^{r-1} \\ y'' &= r(r-1)x^{r-2} \end{aligned}$$

Plugging this into the original equation, and finding the roots of the characteristic polynomial,

$$\begin{aligned} &x^2 \left(r(r-1)x^{r-2}\right) + 4x \left(r x^{r-1}\right) = 2\left(x^r\right) = 0 \\ &\left(r^2 + 3r + 2\right) x^r = 0 \\ &\left(r^2 + 3r + 2\right) = 0 \\ &(r+1)(r+2) = 0 \\ &r = -1,\ -2 \end{aligned}$$

So the complementary solution is

$$y_c(x) = \frac{C_1}{x} + \frac{C_2}{x^2}$$

For the particular solution, we could use variation of parameters, but instead I'm going to use the method of undetermined coefficients. We'll look for particular solutions Y of the form:

$$Y(x) = Ax^5$$

Taking derivatives,

$$\begin{aligned} Y' &= 5Ax^4 \\ Y'' &= 20Ax^3 \end{aligned}$$

Plugging this into the original equation,

$$\begin{aligned} x^2 Y'' + 4xY' + 2Y &= x^5 \\ x^2\left(20Ax^3\right) + 4x\left(5Ax^4\right) + 2\left(Ax^5\right) &= x^5 \\ 42Ax^5 &= x^5 \\ 42A &= 1 \\ A &= \frac{1}{42} \end{aligned}$$

So the particular solution is

$$Y = \frac{1}{42}x^5$$

and the general solution $y = y_c + Y$ is:

$$\boxed{y(x) = \frac{C_1}{x} + \frac{C_2}{x^2} + \frac{1}{42}x^5}$$

Answer to Question 6.
(a) Since this is an Euler equation, we'll look for solutions of the form $y = x^r$. Taking derivatives,

$$\begin{aligned} y &= x^r \\ y' &= rx^{r-1} \\ y'' &= r(r-1)x^{r-2} \end{aligned}$$

Plugging this into the original equation and solving for the roots of the characteristic equation,

$$\begin{aligned} x^2\left(r(r-1)x^{r-2}\right) + x\left(rx^{r-1}\right) + \lambda\left(x^r\right) &= 0 \\ \left(r^2+\lambda\right)x^r &= 0 \\ r^2 + \lambda &= 0 \\ r &= \pm\sqrt{\lambda}i \end{aligned}$$

So the general solution is of the form

$$y = C_1 x^{\sqrt{\lambda}i} + C_2 x^{-\sqrt{\lambda}i}$$

which we can simplify as:

$$\begin{aligned} y &= C_1 x^{\sqrt{\lambda}i} + C_2 x^{-\sqrt{\lambda}i} \\ y &= C_1 e^{i\sqrt{\lambda}\ln(x)} + C_2 e^{-i\sqrt{\lambda}\ln(x)} \end{aligned}$$

and then using Euler's formula,

$$\boxed{y = C_1\cos\left(\sqrt{\lambda}\ln(x)\right) + C_2\sin\left(\sqrt{\lambda}\ln(x)\right)}$$

(b) Plugging in the first boundary condition of $y(1) = 0$,

$$\begin{aligned} y(1) &= C_1\cos\left(\sqrt{\lambda}\ln(1)\right) + C_2\sin\left(\sqrt{\lambda}\ln(1)\right) = 0 \\ y(1) &= C_1\cos(0) + C_2\sin(0) = 0 \\ y(1) &= C_1 = 0 \end{aligned}$$

Using this with the second boundary condition of $y(2) = 0$,

$$y(2) = C_2\sin\left(\sqrt{\lambda}\ln(2)\right) = 0$$

If $C_2 = 0$, we would just have the trivial solution $y = 0$.
So to get nontrivial solutions, we need:

$$\begin{aligned} \sin\left(\sqrt{\lambda}\ln(2)\right) &= 0 \\ \sqrt{\lambda}\ln(2) &= n\pi, \qquad n = 1, 2, 3, \ldots \end{aligned}$$

Therefore the eigenvalues and eigenfunctions are:

$$\boxed{\begin{array}{c}\text{Eigenvalues: } \lambda_n = 1 + \left(\dfrac{n\pi}{\ln(2)}\right)^2 \\ \text{Eigenfunctions: } y_n = \sin\left(\dfrac{n\pi\ln(x)}{\ln(2)}\right) \\ \text{for } \ n = 1, 2, 3, \ldots\end{array}}$$

Answer to Question 7. Since this is an Euler equation, we'll look for solutions of the form $y = x^r$ Taking derivatives,

$$\begin{aligned} y &= x^r \\ y' &= rx^{r-1} \\ y'' &= r(r-1)x^{r-2} \end{aligned}$$

Plugging this into the original equation,

$$\begin{aligned} x^2y'' + \beta y &= 0 \\ x^2\left(r(r-1)x^{r-2}\right) + \beta\left(x^r\right) &= 0 \\ \left(r^2 - r + \beta\right)x^r &= 0 \\ r^2 - r + \beta &= 0 \end{aligned}$$

Using the quadratic formula, our roots are:

$$r = \frac{1 \pm \sqrt{1-4\beta}}{2}$$

To guarantee that all solutions approach zero as $x \to 0$, we will need to make sure that the real part of both roots is positive.

If the roots are complex, then they will have real part $1/2$, and so the solutions will approach zero as $x \to 0$.

If the roots are both real, then we want to make sure that:

$$\begin{aligned} 1 - \sqrt{1-4\beta} &> 0 \\ 1 &> \sqrt{1-4\beta} \\ 1 &> 1 - 4\beta \end{aligned}$$

$$\boxed{\beta > 0}$$