Math 2930 Worksheet

Forced Vibrations
Higher-Order ODEs

Week 9
March 22nd, 2019

Question 1. Find the general solution of the 4th order differential equation:

$$y^{(4)} + 2y'' + y = 0$$

Question 2. Find the general solution of the 6th order differential equation:

$$y^{(6)} - y'' = 0$$

Question 3. Solve the initial value problem:

$$y^{(4)} - 4y''' + 4y'' = 0$$

$$y(1) = -1, \quad y'(1) = 2, \quad y''(1) = 0, \quad y'''(1) = 0$$

Question 4. A damped forced oscillator is described by the equation:

$$u'' + \lambda u' + u = F_0 \sin(\omega t)$$

where $\lambda > 0$.

(a) Find the steady state (*i.e.* particular) solution of this equation.

(b) Find the amplitude of the steady state solution you found in part **(a)**.

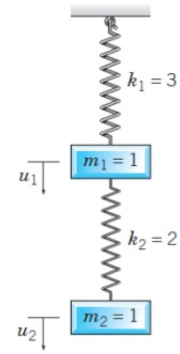
Question 5. Suppose there are two masses m_1 and m_2. m_1 is suspended by a spring hanging from the ceiling, and m_2 is suspended by another spring hanging from m_2 as in the picture below. Their positions u_1 and u_2 satisfy the coupled system of equations:

$$u_1'' + 5u_1 = 2u_2, \qquad u_2'' + 2u_2 = 2u_1 \tag{1}$$

(a) Solving the first equation of (1) for u_2 and substituting into the second equation, we obtain the following fourth-order equation for u_1:

$$u_1^{(4)} + 7u_1'' + 6u_1 = 0 \tag{2}$$

Find the general solution of equation (2)



(b) Suppose that the initial conditions are:

$$u_1(0) = 1, \qquad u_1'(0) = 0, \qquad u_2(0) = 2, \qquad u_2'(0) = 0$$

Use these initial conditions and the first equation of (1) to obtain values for $u_1''(0)$ and $u_1'''(0)$.

(c) Show that the solution of Eq. (2) that satisfies the initial conditions you found in part **(b)** is

$$u_1(t) = \cos(t)$$

(d) Given your $u_1(t)$ from part **(c)**, use this to find $u_2(t)$

Question 6. Consider a horizontal metal beam of length L subject to a vertical load $f(x)$ per unit length. The resulting vertical displacement in the beam $y(x)$ satisfies a differential equation of the form

$$A\frac{d^4y}{dx^4} = f(x)$$

where A is a constant related to Young's modulus and the moment of inertia of the beam. (See picture below).
Suppose that $f(x)$ is a constant k:

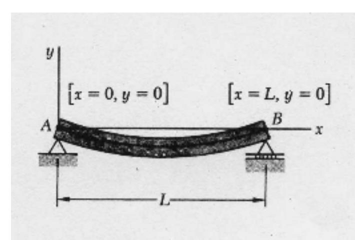
$$A\frac{d^4y}{dx^4} = k$$

(a) Find the general solution of this non-homogeneous fourth-order equation:

For each of the boundary conditions given below, solve for the displacement $y(x)$:

(b) Simply supported at both ends:

$$y(0) = y''(0) = y(L) = y''(L) = 0$$

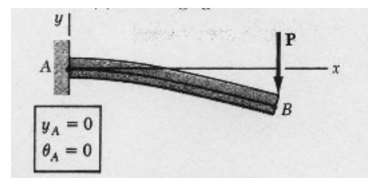


(c) Clamped at both ends:

$$y(0) = y'(0) = y(L) = y'(L) = 0$$

(d) Clamped at $x = 0$, free at $x = L$:

$$y(0) = y'(0) = y''(L) = y'''(L) = 0$$



Answer to Question 1. To find the general solution, we will solve for the roots of the characteristic equation:

$$\begin{aligned} y^{(4)} + 2y'' + y &= 0 \\ r^4 + 2r^2 + 1 &= 0 \\ (r^2+1)^2 &= 0 \\ r &= \pm i,\ \pm i(\text{repeated}) \end{aligned}$$

Since the $\pm i$ roots are repeated, the general solution is:

$$\boxed{y = c_1 \cos(t) + c_2 \sin(t) + c_3 t \cos(t) + c_4 t \sin(t)}$$

Answer to Question 2. To find the general solution, we will solve for the roots of the characteristic equation:

$$\begin{aligned} y^{(6)} - y'' &= 0 \\ r^6 - r^2 &= 0 \\ r^2(r^4-1) &= 0 \\ r^2(r^2+1)(r^2-1) &= 0 \\ r^2(r^2+1)(r+1)(r-1) &= 0 \\ r &= 0,\ 0,\ \pm i,\ 1,\ -1 \end{aligned}$$

So the corresponding general solution is:

$$\boxed{y = c_1 + c_2 t + c_3 \cos(t) + c_4 \sin(t) + c_5 e^t + c_6 e^{-t}}$$

Answer to Question 3. First, we will start by finding the general solution using the roots of the characteristic equation:

$$\begin{aligned} y^{(4)} - 4y''' + 4y'' &= 0 \\ r^4 - 4r^3 + 4r^2 &= 0 \\ r^2(r^2 - 4r + 4) &= 0 \\ r^2(r-2)^2 &= 0 \\ r &= 0, 0, 2, 2 \end{aligned}$$

and the corresponding general solution is:

$$y = c_1 + c_2 t + c_3 e^{2t} + c_4 t e^{2t}$$

Now we will need to use the initial conditions to find c_1, c_2, c_3, and c_4. Taking derivatives,

$$\begin{aligned} y &= c_1 + c_2 t + c_3 e^{2t} + c_4 t e^{2t} \\ y' &= c_2 + 2c_3 e^{2t} + c_4 e^{2t} + 2c_4 t e^{2t} \\ y'' &= 4c_3 e^{2t} + 4c_4 e^{2t} + 4c_4 t e^{2t} \\ y''' &= 8c_3 e^{2t} + 12c_4 e^{2t} + 8c_4 t e^{2t} \end{aligned}$$

Then, plugging in $t = 1$, we get the following system of equations:

$$\begin{aligned}
y(1) &= c_1 + c_2 + c_3e^2 + c_4e^2 && = -1 \\
y'(1) &= c_2 + 2c_3e^2 + 3c_4e^2 && = 2 \\
y''(1) &= 4c_3e^2 + 8c_4e^2 && = 0 \\
y''''(1) &= 8c_3e^2 + 20c_4e^2 && = 0
\end{aligned}$$

The last two equations can be solved to find that $c_3 = c_4 = 0$.
The remaining equations become:

$$\begin{aligned}
c_1 + c_2 &= -1 \\
c_2 &= 2
\end{aligned}$$

which is solved by $c_1 = -1$ and $c_2 = 2$.
Putting this all together, the solution is:

$$\boxed{y = 2t - 3}$$

Answer to Question 4. (a) We can solve for the particular equation using the method of undetermined coefficients.
We will guess a particular solution of the form

$$Y = A\cos(\omega t) + B\sin(\omega t)$$

(As an aside, we do not have to worry about multiplying this by a factor of t, since the roots of the original equation will always be real or complex, but never imaginary. So they will never give homogeneous solutions of just cos() or sin().)

Taking derivatives,

$$\begin{aligned}
Y &= A\cos(\omega t) + B\sin(\omega t) \\
Y' &= -\omega A\sin(\omega t) + \omega B\cos(\omega t) \\
Y'' &= -\omega^2 A\cos(\omega t) - \omega^2 B\cos(\omega t)
\end{aligned}$$

Plugging this into the original equation,

$$\begin{aligned}
Y'' + \lambda Y' + Y &= (A + \lambda\omega B - \omega^2 A)\cos(\omega t) + (B - \lambda\omega A - \omega^2 B)\sin(\omega t) \\
&= F_0\sin(\omega t)
\end{aligned}$$

Comparing like terms, we get the following system of two equations for A and B:

$$\begin{aligned}
A + \lambda\omega B - \omega^2 A &= 0 \\
B - \lambda\omega A - \omega^2 B &= F_0
\end{aligned}$$

Using the first equation to solve for A in terms of B, we get:

$$A = \frac{-\lambda\omega}{1 - \omega^2}B$$

Plugging this into the second equation,

$$\begin{aligned} B + \frac{\lambda^2\omega^2}{1-\omega^2} - \omega^2 B &= F_0 \\ B\left[(1-\omega^2)^2 + \lambda^2\omega^2\right] &= F_0(1-\omega^2) \\ B &= \frac{F_0(1-\omega^2)}{\lambda^2\omega^2 + (1-\omega^2)^2} \end{aligned}$$

which also gives us that

$$A = \frac{-F_0\lambda\omega}{\lambda^2\omega^2 + (1-\omega^2)^2}$$

So the particular solution is:

$$\boxed{Y = \frac{-F_0\lambda\omega}{\lambda^2\omega^2 + (1-\omega^2)^2}\cos(\omega t) + \frac{F_0(1-\omega^2)}{\lambda^2\omega^2 + (1-\omega^2)^2}\sin(\omega t)}$$

(b) The amplitude R of $A\cos(\omega t) + B\sin(\omega t)$ is determined by $R^2 = A^2 + B^2$. Using our answers from part **(a)** , that means:

$$\begin{aligned} R^2 &= A^2 + B^2 \\ R^2 &= \frac{F_0^2(1-\omega^2)^2}{(\lambda^2\omega^2 + (1-\omega^2)^2)^2} + \frac{F_0^2\lambda^2\omega^2}{(\lambda^2\omega^2 + (1-\omega^2)^2)^2} \\ R^2 &= \frac{F_0^2[(1-\omega^2) + \lambda^2\omega^2]}{(\lambda^2\omega^2 + (1-\omega^2)^2)^2} \\ R^2 &= \frac{F_0^2}{\lambda^2\omega^2 + (1-\omega^2)^2} \end{aligned}$$

Taking the square root of both sides,

$$\boxed{R = \frac{F_0}{\sqrt{\lambda^2\omega^2 + (1-\omega^2)^2}}}$$

Answer to Question 5. (a) First, we find the general solution by using the roots of the characteristic equation:

$$\begin{aligned} u_1^{(4)} + 7u_1'' + 6u_1 &= 0 \\ r^4 + 7r^2 + 6 &= 0 \\ (r^2+1)(r^2+6) &= 0 \\ r &= \pm i,\ \pm\sqrt{6}i \end{aligned}$$

The corresponding general solution is:

$$\boxed{u = c_1\cos(t) + c_2\sin(t) + c_3\cos(\sqrt{6}t) + c_4\sin(\sqrt{6}t)}$$

(b) To find the initial conditions, we rearrange the first equation to get:

$$u_1'' = 2u_2 - 5u_1 \tag{3}$$

Plugging in $t = 0$,

$$u_1''(0) = 2u_2(0) - 5u_1(0) = 2(2) - 5(1)$$

$$\boxed{u_1''(0) = -1}$$

To find $u'''(0)$, we will take the derivative of (3) to get:

$$u_1''' = 2u_2' - 5u_1'$$

Again, plugging in $t = 0$,

$$u_1'''(0) = 2u_2'(0) - 5u_1'(0) = 2(0) - 5(0)$$

$$\boxed{u_1'''(0) = 0}$$

(c) Our general solution is:

$$u = c_1 \cos(t) + c_2 \sin(t) + c_3 \cos(\sqrt{6}t) + c_4 \sin(\sqrt{6}t)$$

Taking derivatives,

$$\begin{aligned}
u' &= -c_1 \sin(t) + c_2 \cos(t) - \sqrt{6}c_3 \sin(\sqrt{6}t) + \sqrt{6}c_4 \cos(\sqrt{6}t) \\
u'' &= -c_1 \cos(t) - c_2 \sin(t) - 6c_3 \cos(\sqrt{6}t) - 6c_4 \sin(\sqrt{6}t) \\
u''' &= c_1 \sin(t) - c_2 \cos(t) + 6\sqrt{6}c_3 \sin(\sqrt{6}t) - 6\sqrt{6}c_4 \cos(\sqrt{6}t)
\end{aligned}$$

Now, we want to match the initial conditions:

$$u_1(0) = 1, \quad u_1'(0) = 0, \quad u_1''(0) = -1, \quad u_1'''(0) = 0$$

Plugging in $t = 0$, we get the system of equations:

$$\begin{aligned}
u_1(0) &= c_1 + c_3 &&= 1 \\
u_1'(0) &= c_2 + \sqrt{6}c_4 &&= 0 \\
u_1''(0) &= -c_1 - c_3 &&= -1 \\
u_1'''(0) &= -c_2 - 6\sqrt{6}c_4 &&= 0
\end{aligned}$$

The first and third equations can be solved to get $c_1 = 1$ and $c_3 = 0$.
The second and fourth equations can be solved to find $c_2 = c_4 = 0$.
Putting this all together, our solution is:

$$\boxed{u_1(t) = \cos(t)}$$

(d) Solving for $u_2(t)$,

$$u_1'' + 5u_1 = 2u_2$$
$$u_2 = \frac{u_1'' + 5u_1}{2}$$
$$u_2 = \frac{-\cos(t) + 5\cos(t)}{2}$$
$$\boxed{u_2(t) = 2\cos(t)}$$

Answer to Question 6. (a) We can find the general solution of this non-homogeneous fourth-order equation using the method of undetermined coefficients, but it's actually easier to just integrate four times:

$$y^{(4)} = \frac{k}{A}$$
$$y''' = \frac{kx}{A} + C_1$$
$$y'' = \frac{kx^2}{2A} + C_1x + C_2$$
$$y' = \frac{kx^3}{6A} + C_1x^2 + C_2x + C_3$$
$$\boxed{y = \frac{kx^4}{24A} + C_1x^3 + C_2x^2 + C_3x + C_4}$$

(b) We have to be careful here, we can't just use the derivatives of y we had in part **(a)** , because we kept changing what C_1 meant from line to line. For the next parts, I'll use the following derivatives:

$$y = \frac{kx^4}{24A} + C_1x^3 + C_2x^2 + C_3x + C_4$$
$$y' = \frac{kx^3}{6A} + 3C_1x^2 + 2C_2x + C_3$$
$$y'' = \frac{kx^2}{2A} + 6C_1x + 2C_2$$
$$y''' = \frac{kx}{A} + 6C_1$$

Now, for the boundary conditions:

$$y(0) = y''(0) = y(L) = y''(L) = 0$$

Plugging these in,

$$y(0) = C_4 = 0$$
$$y''(0) = 2C_2 = 0$$
$$y(L) = \frac{kL^4}{24A} + C_1L^3 + C_2L^2 + C_3L + C_4 = 0$$
$$y''(L) = \frac{kL^2}{2A} + 6C_1L + 2C_2 = 0$$

The first two equations give us $C_2 = C_4 = 0$. This simplifies the last two equations:

$$\begin{aligned} \frac{kL^4}{24A} + C_1 L^3 + C_3 L &= 0 \\ \frac{kL^2}{2A} + 6C_1 L &= 0 \end{aligned}$$

We can solve the second equation for

$$C_1 = \frac{-kL}{12A}$$

Plugging this in,

$$\begin{aligned} \frac{kL^4}{24A} - \frac{kL^4}{12A} + C_3 L &= 0 \\ C_3 &= \frac{kL^3}{24A} \end{aligned}$$

Putting it all together, our solution is

$$y = \frac{kx^4}{24A} - \frac{kLx^3}{12A} + \frac{kL^3 x}{24A}$$

which we can simplify as:

$$\boxed{y = \frac{k}{24A}(x^4 - 2Lx^3 + L^3 x)}$$

(c) Plugging in the boundary conditions:

$$y(0) = y'(0) = y(L) = y'(L) = 0$$

we get the following system of equations:

$$\begin{aligned} y(0) &= C_4 = 0 \\ y'(0) &= C_3 = 0 \\ y(L) &= \frac{kL^4}{24A} + C_1 L^3 + C_2 L^2 + C_3 L + C_4 = 0 \\ y'(L) &= \frac{kL^3}{6A} + 3C_1 L^2 + 2C_2 L + C_3 = 0 \end{aligned}$$

Using $C_3 = C_4 = 0$ from the first two equations, we can reduce this to the systemL

$$\begin{aligned} \frac{kL^4}{24A} + C_1 L^3 + C_2 L^2 &= 0 \\ \frac{kL^3}{6A} + 3C_1 L^2 + 2C_2 L &= 0 \end{aligned}$$

Subtracting the first equation from the $2L$ times the second equation,

$$\begin{aligned} \frac{kL^4}{12A} + C_1 L^3 &= 0 \\ C_1 &= \frac{-kL}{12A} \end{aligned}$$

Substituting that back in,

$$\frac{kL^4}{24A} - \frac{2kL^4}{24A} + C_2L^2 = 0$$
$$C_2 = \frac{kL^2}{24A}$$

So putting this all back together,

$$\boxed{y = \frac{k}{24A}(x^4 - 2Lx^3 + L^2x^2)}$$

(d) Plugging in the boundary conditions:

$$y(0) = y'(0) = y''(L) = y'''(L) = 0$$

We get the following system of equations:

$$y(0) = C_4 = 0$$
$$y'(0) = C_3 = 0$$
$$y''(L) = \frac{kL^2}{2A} + 6C_1L + 2C_2 = 0$$
$$y'''(L) = \frac{kL}{A} + 6C_1 = 0$$

We can solve the last equation to get:

$$C_1 = \frac{-kL}{6A}$$

Substituting that in,

$$\frac{kL^2}{2A} - \frac{kL^2}{A} + 2C_2 = 0$$
$$C_2 = \frac{kL^2}{4A}$$

Putting this all back together, our solution is:

$$y = \frac{kx^4}{24A} - \frac{kLx^3}{6A} + \frac{kL^2x^2}{4A}$$

which simplifies to:

$$\boxed{y = \frac{k}{24A}(x^4 - 4Lx^3 + 6L^2x^2)}$$