2nd-Order Homogeneous Equations with Constant Coefficients

Let's go over an example of how to solve an initial value problem for a second-order linear homogeneous equation with constant coefficients.

Example

Suppose we are given the following problem:

$$5y'' + 2y' + y = 0$$
$$y(0) = 2, \quad y'(0) = 1$$

To find the general solution we will look for solutions of the form

$$y = e^{rt}$$

(If you want to be fancy, the technical term for this is an *Ansatz*, but that's just a German word that can be translated in this context as "guess".)

Anyway, we can first compute that the derivatives are:

$$y = e^{rt}$$
$$y' = re^{rt}$$
$$y'' = r^2 e^{rt}$$

Plugging these into the original differential equation,

$$5y'' + 2y' + y = 0$$
$$5r^2 e^{rt} + 2re^{rt} + e^{rt} = 0$$

Factoring out an e^{rt},

$$\left(5r^2 + 2r + 1\right) e^{rt} = 0$$

Since we want to find a value of r that works for all t, and e^{rt} is never zero, we can divide out e^{rt} from both sides, giving us the *characteristic polynomial*

$$5r^2 + 2r + 1 = 0$$

To find the roots of this polynomial, we will have to use the quadratic formula:

$$r = \frac{-2 \pm \sqrt{4 - 4(5)}}{2(5)}$$

Simplifying,

$$r = \frac{-2 \pm \sqrt{4-20}}{10}$$
$$r = \frac{-2 \pm \sqrt{-16}}{10} = \frac{-2 \pm 4i}{10}$$
$$r = \frac{-1}{5} \pm \frac{2i}{5}$$

Since these roots are complex, our general solution is:

$$\boxed{y(t) = c_1 e^{-t/5} \cos\left(\frac{2t}{5}\right) + c_2 e^{-t/5} \sin\left(\frac{2t}{5}\right)}$$

Now we have to plug in the initial values to solve for c_1 and c_2.

First, let's calculate the derivative of our general solution. Using both the product and chain rules,

$$y'(t) = -\frac{c_1}{5} e^{-t} \cos\left(\frac{2t}{5}\right) - \frac{2c_1}{5} e^{-t/5} \sin\left(\frac{2t}{5}\right) - \frac{c_2}{5} e^{-t/5} \sin\left(\frac{2t}{5}\right) + \frac{2c_2}{5} e^{-t/5} \cos\left(\frac{2t}{5}\right)$$

Plugging in the initial conditions for both y and y', we get the following two equations

$$y(0) = c_1 = 2$$
$$y'(0) = -\frac{c_1}{5} + \frac{2c_2}{5} = 1$$

In general, we will get two linear equations in our two unknowns c_1 and c_2.

Plugging in $c_1 = 2$ into the second equation, and then multiplying both sides by 5,

$$-2 + 2c_2 = 5$$
$$c_2 = \frac{7}{2}$$

So our constants are $c_1 = 2$ and $c_2 = \frac{7}{2}$. Plugging this into our general solution, we find the solution to our initila value problem to be

$$\boxed{y(t) = 2e^{-t/5} \cos\left(\frac{2t}{5}\right) + \frac{7}{2} e^{-t/5} \sin\left(\frac{2t}{5}\right)}$$