Math 2930 Discussion Notes
Exact Equations

Week 4
February 15, 2019

1 Exact Equations Example

Let's start today with the example:

$$\big(y\cos(x) - 2y^2\big) + \big(\sin(x) - 4xy\big)\frac{dy}{dx} = 0 \tag{1}$$

We can check that this equation is:

- not linear
- not separable
- not homogeneous

so we won't be able to use any of those techniques. Instead, this equation is what is called an *exact* equation. The idea here is that we will look for solutions in the form

$$\psi(x, y) = C$$

where ψ is some yet-to-be-determined function of x and y. If we take the *total* derivative of ψ with respect to x, then according to the chain rule,

$$\frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{dy}{dx} = 0 \tag{2}$$

So now we will try to choose ψ in such a way that this matches our original equation (1). This means we are looking for a function ψ with:

$$\frac{\partial\psi}{\partial x} = y\cos(x) - 2y^2 \qquad \text{and} \qquad \frac{\partial\psi}{\partial y} = \sin(x) - 4xy \tag{3}$$

In general, for equations of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0$$

we look for a function ψ whose partial derivatives are:

$$\frac{\partial\psi}{\partial x} = M(x, y) \qquad \text{and} \qquad \frac{\partial\psi}{\partial y} = N(x, y)$$

But how do we know that we can find a function $\psi(x, y)$ exists? Well, let's assume we did have such a function $\psi(x, y)$. Then multivariable calculus would tell us that its mixed partial derivatives should always be equal, *i.e.*

$$\psi_{xy} = \psi_{yx}$$

which would mean that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{4}$$

so if we had a function ψ, then M and N would have to satisfy this condition. Equations where this condition on M and N is satisfied are called *exact* equations. It actually turns out that whenever (4) is satisfied, we can always find an appropriate function ψ that satisfies:

$$\frac{\partial \psi}{\partial x} = M(x, y) \qquad \text{and} \qquad \frac{\partial \psi}{\partial y} = N(x, y)$$

So for our example above, we compute

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left[y\cos(x) - 2y^2\right] = \cos(x) - 4y$$

and

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left[\sin(x) - 4xy\right] = \cos(x) - 4y$$

so Equation (1) is exact.

Now, we want to choose our function ψ so that Equation (2) matches Equation (1). This gives us two equations:

$$\frac{\partial \psi}{\partial x} = y\cos(x) - 2y^2 \qquad \text{and} \qquad \frac{\partial \psi}{\partial y} = \sin(x) - 4xy$$

Let's start with the first equation. If we integrate both sides with respect to x, we get

$$\int \frac{\partial \psi}{\partial x} dx = \int \left(y\cos(x) - 2y^2\right) dx$$
$$\psi = y\sin(x) - 2xy^2 + f(y)$$

You are probably used to adding a $+C$ at the end of any integrals, but since we are undoing a *partial* derivative here, the appropriate term is instead a function $f(y)$ of y only instead of a constant. If we repeat the same process with the other partial derivative, we get:

$$\int \frac{\partial \psi}{\partial y} dy = \int \left(\sin(x) - 4xy\right) dy$$
$$\psi = y\sin(x) - 2xy^2 + g(x)$$

If we compare these two expressions for $\psi(x, y)$ we see that they are equivalent in the case when $f(y) = g(x) = 0$. In this case we have

$$\psi(x, y) = y\sin(x) - 2xy^2$$

so the solution to our original Equation (1) is

$$\boxed{y\sin(x) - 2xy^2 = C}$$