Math 2930 Discussion Notes

Fourier Series
Tabular Integration by Parts

Week 10
October 26th, 2017

1 A Linear Algebra Perspective on Fourier Series

The idea of *Fourier series* is that given a periodic function $f(x)$ of period $2L$, we want to represent it as a sum of sine and cosine functions also of period $2L$:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

I think that for some people it's useful to think of Fourier series in terms of the idea of basis vectors from multivariable calculus/linear algebra.

For example, any three-dimensional vector $\mathbf{v}$ can be broken down into three components:

$$\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

There are a lot of different notations for this idea, based on context. You may see this as any one of:

$$\mathbf{v} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}} = a\hat{\mathbf{x}} + b\hat{\mathbf{y}} + c\hat{\mathbf{z}} = a\mathbf{e}_1 + b\mathbf{e}_2 + c\mathbf{e}_3$$

depending on what class you are in.

The idea of Fourier series is to turn this idea of representing arbitrary *vectors* in terms of a set of *basis vectors* to representing *functions* in terms of a set of *basis functions*.

In case this idea sounds a little far-fetched, I want to point out that you've already seen this with *Taylor series*. With Taylor series, we break down a given (infinitely differentiable) function into coefficients times powers of x:

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ... = \sum_{n=0}^{\infty} a_n x^n$$

So instead of our basis being the set:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

we now have the set of *basis functions*:

$$\{1, x, x^2, x^3, ...\}$$

Although there are now infinitely many elements in our basis, many parts of the analogy still hold.

So with Fourier series, the idea is to break our periodic function into sine and cosine components as follows:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

So here our corresponding basis functions would be:

$$\left\{1, \cos\left(\frac{\pi x}{L}\right), \sin\left(\frac{\pi x}{L}\right), \cos\left(\frac{2\pi x}{L}\right), \sin\left(\frac{2\pi x}{L}\right), ...\right\}$$

So for a given periodic function f, how do we find the coefficients for these basis elements? This can be found by computing the following integrals:

$$\begin{aligned} a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)dx \\ a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \end{aligned}$$

2 Tabular Integration by Parts

Now, a quick note on actually computing these integrals. If say, $f(x) = x^2$ and $L = \pi$, then we are left computing integrals like:

$$a_n = \int_{-\pi}^{\pi} x^2 \cos(nx)dx$$

Computing this integral requires using integration by parts twice. But doing integration by parts once gets you an integral that looks like $\int x\sin(nx)dx$, which needs yet another application of integration by parts. Keeping track of all this by doing $U = ...$, $dV = ...$ every step can be very time-consuming and also error-prone if you have to do it multiple times in a row.

A more systematic way of doing this is called *tabular integration by parts*. You start by writing down a table where the S column alternates between $+$ and $-$. In the D column, you start with "U" and keep taking derivatives. In the I column, you start with "V" and keep taking integrals:

S	D	I
$+$	x^2	$\cos(nx)$
$-$	$2x$	$\frac{1}{n}\sin(nx)$
$+$	2	$\frac{-1}{n^2}\cos(nx)$
$-$	0	$\frac{-1}{n^3}\sin(nx)$

We stop at this point, since any further entries in the D column will always be zero.

Then, to compute $\int x^2\cos(nx)dx$, we read off the entries as follows:

S	D	I
$+$	x^2	$\cos(nx)$
$-$	$2x$	$\frac{1}{n}\sin(nx)$
$+$	2	$\frac{-1}{n^2}\cos(nx)$
$-$	0	$\frac{-1}{n^3}\sin(nx)$

Which we read off as:

$$\int x^2\cos(nx)dx = (+)(x^2)\left(\frac{1}{n}\sin(nx)\right) + (-)(2x)\left(\frac{-1}{n^2}\cos(nx)\right) + (+)(2)\left(\frac{-1}{n^3}\sin(nx)\right) + C$$

which cleans up to:

$$\int x^2\cos(nx)dx = \frac{x^2}{n}\sin(nx) + \frac{2x}{n^2}\cos(nx) + \frac{-2}{n^3}\sin(nx) + C$$

So this is a much cleaner and less error-prone way of doing these integrals that require repeated integration by parts (which show up a lot in Fourier series problems).