Math 2930 Worksheet
Introduction to Differential Equations

Week 2
February 1st, 2019

Learning Goals

- Solve linear first order ODEs analytically.
- Solve separable first order ODEs analytically.

Questions

Question 1. Solve the following differential equations:
(a) $y' - 2y = 3e^t$

(b) $\dfrac{dy}{dx} = \dfrac{x - e^{-x}}{y + e^y}$

(c) $t^3 y' + 4t^2 y = e^{-t}$

Question 2. In the homework you were asked to analyze the behavior of the solutions of

$$y' = -2 + t - y$$

as $t \to \infty$ using a sketch of the direction field. Now solve this differential equation and use the analytic solution to determine the behavior of its solutions as $t \to \infty$.

Question 3. Consider the initial value problem:

$$y' - 3y = 3t + e^{2t}, \qquad y(0) = y_0$$

Find the value of y_0 that separates solutions that grow positively as $t \to \infty$ from those that grow negatively. How does the solution that corresponds to this critical value of y_0 behave as $t \to \infty$?

Question 4. A method of radioactive dating involves a substance with a half-life of T years. Assuming the radioactive substance decays according to the law

$$Q' = -rQ$$

(a) Find an expression for r in terms of T .

(b) A sample has current contents of the radioactive substance equal to 10% of the amount that was originally present. Find the age of this sample in terms of T.

Question 5. A rocket sled having an initial speed of v_0 is slowed by a channel of water. Assume that during the braking process, the acceleration $a = dv/dt$ is given by

$$a = -\mu v^2$$

where v is the velocity and μ is a constant.

(a) Find the time it takes for the sled to slow down to speed $v_0/2$ (half its original speed).

(b) If it requires a distance L to slow the sled to speed $v_0/2$, determine an expression for μ in terms of v_0 and L.
Hint: Use the relation

$$\frac{dv}{dt} = v\frac{dv}{dx}$$

Question 6. For the following problems:

- Construct a first-order linear differential equation whose solutions have the required behavior as $t \to \infty$
- Solve your equation and confirm that the solutions do indeed have the specified property

(*Hint*: The equation in Question 2 had a property like this)

(a) All solutions have the limit 3 as $t \to \infty$

(b) All solutions are asymptotic to the line $y = 2t + 1$ as $t \to \infty$

(c) All solutions approach the curve $y = 4 - t^2$ as $t \to \infty$

Solutions

Answer to Question 1.
(a) $y' - 2y = 3e^t$
Since this is a first order equation, we will need to use integrating factors. We multiply both sides by an unknown function $\mu(t)$:

$$\mu y' - 2\mu y = 3e^t \mu$$

Now, in order for the left hand side to resemble a product rule, we need $\mu(t)$ to satisfy:

$$\frac{d\mu}{dt} = -2\mu$$

which we can solve for $\mu(t)$ by separating variables and integrating:

$$\begin{aligned} \int \frac{d\mu}{\mu} &= -2 \int dt \\ \ln(\mu) &= -2t \\ \mu &= e^{-2t} \end{aligned}$$

(You might noticed that I didn't bother adding the $+C$ at the end of the integral. I can get away with this here because with integrating factors, we only need *a* function to multiply both sides of the equation with, so just using $C = 0$ is fine. But later on, I won't be able to get away with dropping the $+C$ when integrating to solve for $y(t)$, since I will need that term to match the initial condition.)

So multiplying both sides of our original equation by e^{-2t} we get:

$$\begin{aligned} e^{-2t}y' - 2e^{-2t}y &= 3e^t e^{-2t} \\ &= 3e^{-t} \end{aligned}$$

According to the product rule, this is

$$\left(e^{-2t}y\right)' = 3e^{-t}$$

Integrating both sides,

$$\begin{aligned} \int \left(e^{-2t}y\right)' dt &= \int 3e^{-t} dt \\ e^{-2t}y &= -3e^{-t} + C \end{aligned}$$

and then solving for y,

$$\boxed{y = -3e^t + Ce^{2t}}$$

(b) $\dfrac{dy}{dx} = \dfrac{x - e^{-x}}{y + e^{y}}$

This equation is separable, so first we separate the x and y terms:

$$(y + e^y)\,dy = \left(x - e^{-x}\right)dx$$

then integrating both sides,

$$\int (y + e^y)\,dy = \int \left(x - e^{-x}\right)dx$$

$$\boxed{\frac{y^2}{2} + e^y = \frac{x^2}{2} + e^{-x} + C}$$

which we could also simplify as:

$$\boxed{y^2 - x^2 + 2\left(e^y - e^x\right) = C}$$

(c) $t^3 y' + 4t^2 y = e^{-t}$

First, let's make things a little simpler by dividing t^3 from both sides:

$$y' + \frac{4}{t}y = \frac{e^{-t}}{t^3} \tag{1}$$

Since this is a first-order linear equation, we will need to multiply both sides by an integrating factor $\mu(t)$,

$$\mu y' + \frac{4\mu}{t}y = \frac{e^{-t}\mu}{t^3}$$

In order to make the left hand side look like a product rule, μ will have to satisfy the equation

$$\frac{d\mu}{dt} = \frac{4\mu}{t}$$

which is a separable equation. Separating variables and integrating,

$$\int \frac{d\mu}{\mu} = \int \frac{4}{t}dt$$

$$\ln(\mu) = 4\ln(t)$$

$$\mu = e^{4\ln(t)} = t^4$$

So plugging in $\mu = t^4$, we get

$$t^4 y' + 4t^3 y = te^{-t}$$

using the product rule in reverse, we get

$$\left(t^4 y\right)' = te^{-t}$$

integrating both sides (we will have to use integration by parts on the right hand side),

$$\int \left(t^4 y\right)' dt = \int te^{-t}dt$$

$$t^4 y = -te^{-t} - e^{-t} + C$$

then solving for y,

$$\boxed{y = -\frac{e^{-t}}{t^3} - \frac{e^{-t}}{t^4} + \frac{C}{t^4}}$$

Answer to Question 2. $y' = -2 + t - y$
This is a first order linear equation, so first we will rearrange into a more standard form of

$$y' + y = -2 + t$$

Then we multiply both sides by the integrating factor e^t,

$$e^t y' + e^t y = -2e^t + te^t$$

Using the product rule in reverse,

$$\left(e^t y\right)' = -2e^t + te^t$$

then integrating both sides,

$$\begin{aligned} e^t y &= -2\int e^t dt + \int te^t dt \\ e^t y &= -2e^t + (t-1)e^t + C \\ e^t y &= (t-3)e^t + C \end{aligned}$$

and solving for y,

$$\boxed{y = (t-3) + Ce^{-t}}$$

we can now see that as $t \to \infty$, the Ce^{-t} term will approach zero very rapidly, leaving

$$\boxed{y \to t - 3 \qquad \text{as} \quad t \to \infty}$$

Answer to Question 3. $y' - 3y = 3t + e^{2t}$
This is a linear equation, so we multiply both sides by the integrating factor e^{-3t}, getting

$$e^{-3t} y' - 3e^{-3t} y = 3te^{-3t} + e^{-t}$$

using the product rule in reverse,

$$\left(e^{-3t} y\right)' = 3te^{-3t} + e^{-t}$$

Integrating both sides,

$$\begin{aligned} \int \left(e^{-3t} y\right)' dt &= \int 3te^{-3t} + e^{-t} dt \\ e^{-3t} y &= \frac{-1}{3} e^{-3t}(3t+1) - e^{-t} + C \\ y &= -t - \frac{1}{3} - e^{2t} + Ce^{3t} \end{aligned}$$

then plugging in the initial condition $y(0) = y_0$ and solving for C,

$$\begin{aligned} y_0 &= -0 - \frac{1}{3} - 1 + C \\ C &= y_0 + \frac{4}{3} \end{aligned}$$

so our solution is

$$\boxed{-t - \frac{1}{3} - e^{-2t} + \left(y_0 + \frac{4}{3}\right) e^{3t}}$$

We now see that as $t \to \infty$, the e^{3t} term will dominate the others, and thus the long-term behavior of the solution will be dominated by the sign of its coefficient. It follows that the critical value is:

$$\boxed{y_0 = -\frac{4}{3}}$$

If $y_0 + \frac{4}{3} < 0$, i.e. $y_0 < -\frac{4}{3}$, then y will grow negatively as $t \to \infty$.

Similarly, if $y_0 > -\frac{4}{3}$, then y will grow positively as $t \to \infty$.

At exactly the critical value of $y_0 = -\frac{4}{3}$, then the e^{-2t} term will dominate, so the solution will approach $-\infty$ as $t \to \infty$.

Answer to Question 4.
(a) First we will solve the differential equation

$$Q' = -rQ$$

Separating variables and integrating,

$$\begin{aligned} \int \frac{dQ}{Q} &= \int -r dt \\ \ln(Q) &= -rt + C \\ Q &= Ke^{-rt} \end{aligned}$$

We will define Q_0 to be the initial amount of the substance. Mathematically speaking, this means the initial condition for our differential equation is $Q(0) = Q_0$, which we plug in to get:

$$Q(t) = Q_0 e^{-rt}$$

The half-life T of the substance is defined as the time it takes for half of the initial substance to decay, that is

$$Q(T) = \frac{Q_0}{2}$$

plugging this into our equation for $Q(t)$, and then solving for r,

$$\begin{aligned} Q(T) = \frac{Q_0}{2} &= Q_0 e^{-rT} \\ \frac{1}{2} &= e^{-rT} \\ rT &= \ln(2) \end{aligned}$$

$$\boxed{r = \frac{\ln(2)}{T}}$$

(b) First we will define τ to be the age of the sample. Using our formula for $Q(t)$ above, we get

$$Q(\tau) = Q_0 e^{-r\tau} = \frac{Q_0}{10}$$

and then using our answer from part **(a)** ,

$$Q_0 e^{\frac{-\ln(2)\tau}{T}} = \frac{Q_0}{10}$$

Solving for τ,

$$e^{\frac{-\ln(2)\tau}{T}} = \frac{1}{10}$$
$$\frac{\ln(2)\tau}{T} = \ln(10)$$

$$\boxed{\tau = \frac{\ln(10)}{\ln(2)} T = \log_2(10) T}$$

Answer to Question 5.
(a) First, we need to solve the differential equation for $v(t)$

$$\frac{dv}{dt} = -\mu v^2$$

Separating variables and integrating,

$$\int \frac{1}{v^2} dv = -\mu \int dt$$
$$-\frac{1}{v} = -\mu t + C$$

solving for v,

$$v(t) = \frac{1}{\mu t - C}$$

Now we plug in the initial condition $v(0) = v_0$ and solve for C,

$$v(0) = v_0 = \frac{1}{-C}$$
$$C = \frac{-1}{v_0}$$

using this equation for C in our equation for $v(t)$ and then simplifying,

$$v(t) = \frac{1}{\frac{1}{v_0} + \mu t}$$
$$v(t) = \frac{v_0}{1 + v_0 \mu t}$$

Now to find the time it takes for the sled to slow down to speed $v_0/2$, we set

$$\frac{v_0}{2} = \frac{v_0}{1 + v_0 \mu t}$$

and solve for t:

$$2 = 1 + v_0 \mu t$$

$$\boxed{t = \frac{1}{v_0 \mu}}$$

(b)

Approach 1: Solve for $v(x)$

Using the hint,

$$\frac{dv}{dt} = v\frac{dv}{dx} = -\mu v^2$$
$$\frac{dv}{dx} = -\mu v$$

Which we can solve to find v as a function of x (different from v as a function of t!!)

$$v(x) = v_0 e^{-\mu x}$$

then we plug in $x = L$, and solve for μ as follows:

$$v(L) = \frac{v_0}{2} = v_0 e^{-\mu L}$$
$$\frac{1}{2} = e^{-\mu L}$$
$$\ln(2) = \mu L$$

$$\boxed{\mu = \frac{\ln(2)}{L}}$$

Approach 2: Solve for $x(t)$ and plug in answer from (a)

We can recover the distance $x(t)$ traveled by time t through integrating $v(t)$:

$$x = \int 0^t v(t)dt = \int \frac{v_0}{1 + v_0\mu t}dt$$
$$x = \frac{1}{\mu}\ln\left(1 + v_0\mu t\right)$$

now plugging in $x = L$, and $t = \frac{1}{v_0\mu}$ from **(a)** , we get

$$L = \frac{1}{\mu}\ln\left(1 + v_0\mu\frac{1}{v_0\mu}\right) = \frac{\ln(2)}{\mu}$$

which we can solve for μ to get

$$\boxed{\mu = \frac{\ln(2)}{L}}$$

Answer to Question 6. (a) One possible form for our solutions could be:

$$y = 3 + y_0 e^{-t}$$

Now we want to reverse engineer a differential equation from this. Differentiating both sides,

$$\frac{dy}{dt} = -y_0 e^{-t}$$

we can also check that

$$3 - y = y_0 e^{-t}$$

Combining these two, we get that one possible differential equation is:

$$\boxed{\frac{dy}{dt} = 3 - y}$$

Now, to solve this equation, we first write it in the form

$$y' + y = 3$$

Multiplying both sides by the integrating factor e^t,

$$\begin{aligned} e^t y' + e^t y &= 3e^t \\ \left(e^t y\right)' &= 3e^t \\ e^t y &= 3 \int e^t dt \\ e^t y &= 3e^t + C \\ y &= 3 + Ce^{-t} \end{aligned}$$

as desired.

(b) One possible form for our solutions could be:

$$y = (2t + 1) + y_0 e^{-t}$$

Differentiating both sides,

$$\frac{dy}{dt} = 2 - y_0 e^{-t}$$

We can also check that

$$\frac{dy}{dt} = -y + 3 + 2t$$

We combine both of these to get a possible differential equation:

$$\boxed{\frac{dy}{dt} = -y + 3 + 2t}$$

Now, to solve this equation, we write it as

$$y' + y = 3 + 2t$$

using the integrating factor of e^t,

$$\begin{aligned}
e^t y' + e^t y &= 3e^t + 2te^t \\
\left(e^t y\right)' &= 3e^t + 2te^t \\
e^t y &= 3\int e^t dt + 2\int te^t dt \\
e^t y &= 3e^t + 2(t-1)e^t + C \\
y &= 2t + 1 + Ce^{-t}
\end{aligned}$$

as desired.

(c) One possible form for our solutions could be:

$$y = 4 - t^2 + y_0 e^{-t}$$

Differentiating both sides,

$$\frac{dy}{dt} = -2t - y_0 e^{-t}$$

We also find that

$$-y - t^2 - 2t + 4 = -2t - t_0 e^{-t}$$

which gives us a differential equation of

$$\boxed{\frac{dy}{dt} = -y - t^2 - 2t + 4}$$

To solve this equation, we write it as

$$y' + y = -t^2 - 2t + 4$$

then using the integrating factor of e^t,

$$\begin{aligned}
e^t y' + e^t y &= -t^2 e^t - 2te^t + 4e^t \\
\left(e^t y\right)' &= -t^2 e^t - 2te^t + 4e^t \\
e^t y &= -\int t^2 e^t dt - 2\int te^t dt + 4\int e^t dt \\
e^t y &= -(t^2 - 2t + 2)e^t - 2(t-1)e^t + 4e^t + C \\
y &= -(t^2 - 2t + 2) - 2(t-1) + 4 + Ce^{-t} \\
y &= 4 - t^2 + Ce^{-t}
\end{aligned}$$

as desired.