Math 2930 Worksheet
Introduction to Differential Equations

Week 1
January 25, 2019

What is a Differential Equation and what are Solutions?

A **differential equation** is an equation that relates an unknown function to its derivative(s). Suppose $y = y(t)$ is some unknown function, then a differential equation would express the rate of change, $\frac{dy}{dt}$, in terms of y and/or t. For example, all of the following are differential equations.

$$\frac{dP}{dt} = kP, \qquad \frac{dy}{dt} = y + 2t, \qquad \frac{dy}{dx} = x^2 + 5, \qquad \frac{dx}{dt} = \frac{6x - 2}{tx}, \qquad \frac{dy}{dt} = \frac{y^2 - 1}{t^2 + 2t}$$

In particular, these are all examples of *first-order* differential equations because only the first derivative appears in the equation. Given a differential equation for some unknown function, **solutions** to this differential equation are *functions* that satisfy the equation.

Question 1.

- Is the function $y = 1 + t$ a solution to the differential equation $\frac{dy}{dt} = \frac{y^2-1}{t^2+2t}$? Why or why not?
- How about the function $y = 1 + 2t$?
- How about $y = 1$?

Be able to explain your reasoning.

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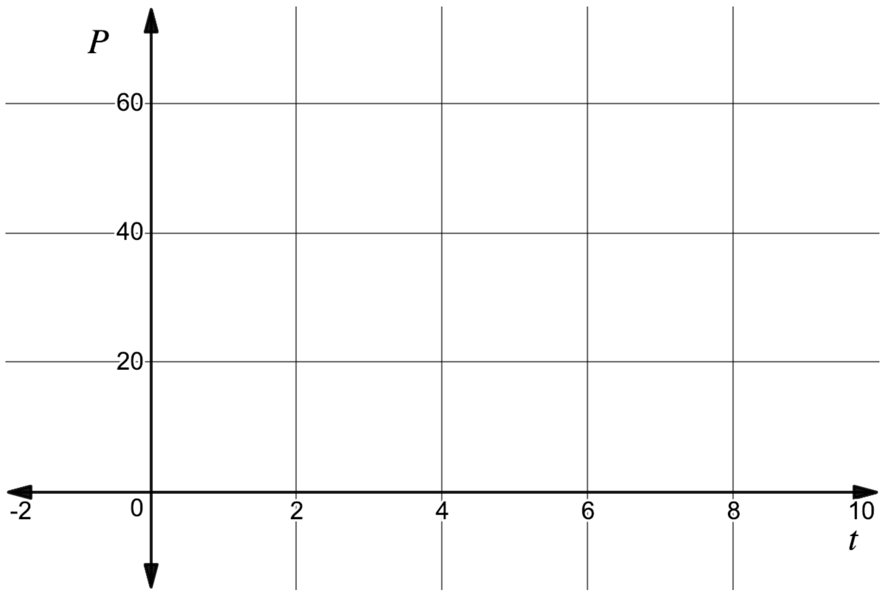
Question 2. Differential equations are often used to model population growth, say of the number of fish in Lake Cayuga as a function of time. Let's simplify the situation by making the following assumptions:

- There is only one species of fish.
- The species has been in Lake Cayuga for some time prior to what we call $t = 0$.
- The species has access to unlimited resources (*e.g.* food, space, water).
- The species reproduces continuously.

Given these assumptions, sketch three possible graphs of population versus time: one starting at $P = 10$, one starting at $P = 20$, and the third starting at $P = 30$.



(a) For your graph starting with $P = 10$, how does the slope vary as time increases? Explain.

(b) For a set P value, say $P = 30$, how do the slopes vary across the three graphs you drew?

Question 3. The situation in Question 2 can be modeled with a differential equation of the form $\frac{dP}{dt} = \textit{something}$. Here are some possible equations someone might use to try and model this situation. For each one, come up with reasons for why they do or don't accurately model the problem.

- $\frac{dP}{dt} = (t+1)^2$

- $\frac{dP}{dt} = 2P$

- $\frac{dP}{dt} = 5e^t$

- $\frac{dP}{dt} = 3t$

- $\frac{dP}{dt} = P^2 + 1$

Question 4. Below are two systems of differential equations. In both of these systems, x and y refer to the population of two different species at time t. Which system describes a situation where the two species compete? Which system describes cooperative species? Explain your reasoning.

$$(i) \quad \frac{dx}{dt} = -5x + 2xy, \qquad \frac{dy}{dt} = -4y + 3xy,$$

$$(ii) \quad \frac{dx}{dt} = x - 2xy \qquad \frac{dy}{dt} = 2y - xy$$

Question 5. Consider the differential equation

$$\frac{dy}{dt} = 0.5y(2 + y)(y - 8)$$

which has been created to provide predictions about the future population of rabbits over time.

- For what values of y is $y(t)$ increasing? Explain your reasoning.
- For what values of y is $y(t)$ decreasing? Explain your reasoning.
- For what values of y is $\frac{dy}{dt}$ neither positive nor negative? What does this imply about the solution function $y(t)$?

Question 6. The differential equation

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y$$

where r and K are positive constants, is an example of what is known as a *logistic equation.* These are usually used to model the population y of some species in an environment with limited resources as a function of time t.

(a) If we start with some initial population $y(0) > 0$, what can you say about the qualitative behavior of $y(t)$ will be as t increases? (By qualitative I mean that you should try to explain using words instead of numbers or equations where possible, but still try to be specific.)

Hint: Does y approach a limit as $t \to \infty$? If so, what is this limiting value? How does this depend on r and K, if at all?

(b) Now suppose we introduce some sort of predation by a fixed number of predators. Then we could model this system with:

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y - Ey$$

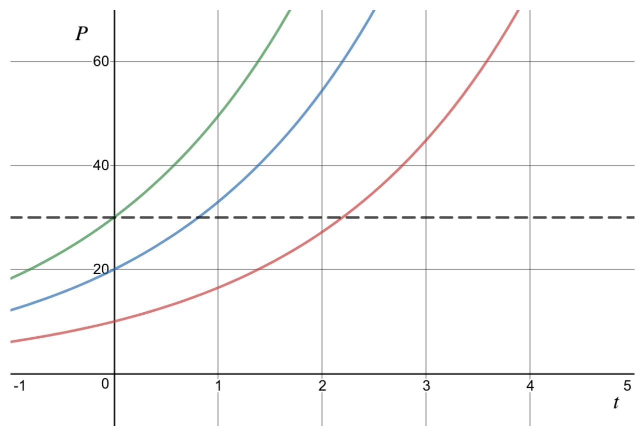
where E is another positive constant. What can you say about the qualitative behavior of $y(t)$ this time? As $t \to \infty$, does the population still approach a non-zero value or is it driven to extinction (i.e. $y \to 0$)? How does this depend on E, if at all?

Answer to Question 1. To check whether these are solutions, we plug the given functions $y(t)$ into both sides of the differential equation and check that they are equal.
For $y = 1 + t$, we get $\frac{dy}{dt} = \frac{d}{dt}\left[1 + t\right] = 1$ and

$$\frac{y^2 - 1}{t^2 + 2t} = \frac{(1+t)^2 - 1}{t^2 + 2t} = \frac{t^2 + 2t}{t^2 + 2t} = 1$$

since these two quantities are equal, $y(t) = 1 + t$ is a solution of the differential equation.
For $y = 1 + 2t$, we get $\frac{dy}{dt} = 2$, but $\frac{y^2-1}{t^2+2t} = \frac{4t^2+4t}{t^2+2t} \neq 2$, so this is not a solution.
For $y = 1$, we get $\frac{dy}{dt} = 0$ and $\frac{y^2-1}{t^2+2t} = 0$, so this is another solution.

Answer to Question 2.
Your graphs could look something like this:



(a) For the graph starting with $P = 10$ (shown in red), the slope increases as time increases. This makes sense, since more fish implies more reproduction, which implies a larger growth rate. Visually, this means that your graph should be concave up.

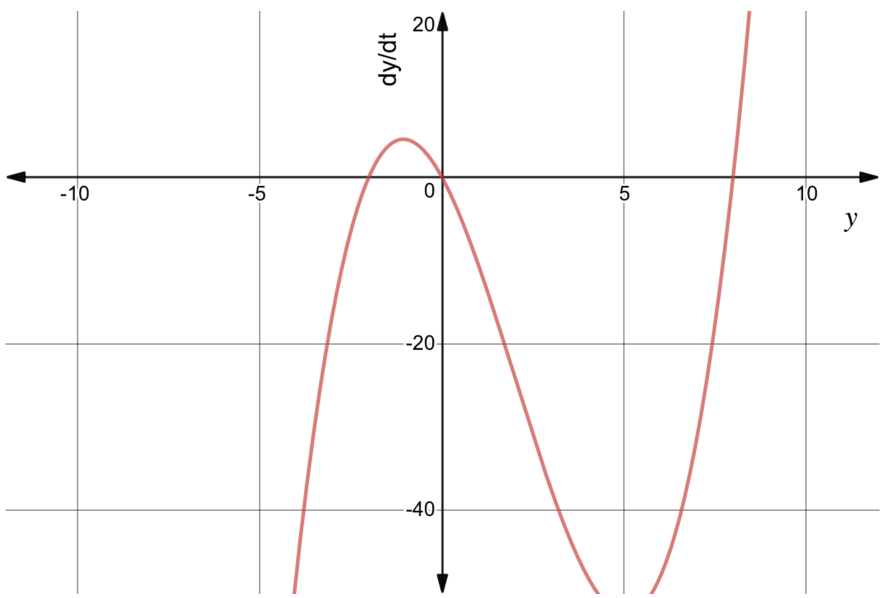
(b) For a set P value, the slope stays the same across the three graphs. This is because the slope of the graph is the rate of change $\frac{dP}{dt}$. Since the growth rate of the population depends only on the number of fish, not the time of day, it should be the same on all three graphs. Visually this means that the slope of the graphs should be the same at all the 3 points where the horizontal dashed line intersects the graphs.

Answer to Question 3.
The rate of change $\frac{dP}{dt}$ should be a function of only the size of the population P, and not on the time t. For similar reasons to (2b), the rate of change is determined by the number of fish there are right now, not what time of day it is. This rules out all but the 2nd and 5th options. Note: When they first see this problem, many students are tempted to put $\frac{dP}{dt}$ = something involving t, because the rate is increasing in time. But since P is itself a function of t, this means $\frac{dP}{dt}$ depends *implicitly* on the value of t, so $\frac{dP}{dt}$ can still be increasing in time even without depending *explicitly* on it.
We can also rule out the 5th option using a sanity check of plugging in $P = 0$. Since $P = 0$ is the case where there are no fish, any reasonable model would also necessarily have $\frac{dP}{dt} = 0$, otherwise fish are being created out of nowhere. Since the fifth model would instead say $\frac{dP}{dt} = 1 > 0$, this model does not make sense.

$\frac{dP}{dt}$ should really depend linearly on P since increasing the number of fish by a constant factor would result in increasing reproduction by that exact same constant factor. I.e. if you have three as many fish parents you should have three times as many fish babies. This leaves $\frac{dP}{dt} = 2P$ as the only good model.

Answer to Question 4.
Here the terms that correspond to the interaction of the two species are those with *both* x and y. Whether the interactions are cooperative or competitive depends on the sign of the xy term.
Since (i) has positive coefficients in front of the xy terms, that means increasing the populations of both x and y have a positive effect on the rate at which both populations grow. This would correspond to the case of two cooperative species.
Similarly, since (ii) has negative coefficients in front of the xy terms, it means that increasing the numbers of x or y would have a negative effect on the rate of change of both species. This would correspond to the case of two competitive species.

Answer to Question 5.
To find where $y(t)$ is increasing or decreasing, we will need to find where $\frac{dy}{dt}$ is positive or negative. We can easily see this by looking at a graph of $\frac{dy}{dt}$ versus y. (Note that this is *not* the same as a graph of y versus t!)



If we are doing this by hand and don't have access to any sort of graphing tool, we can figure out the sign of $\frac{dy}{dt}$ by considering the signs of each of the three terms: $(0.5y)$, $(2 + y)$, and $(y - 8)$.

(a) $y(t)$ is increasing where $\frac{dy}{dt}$ is positive. This happens when $-2 < y < 0$ and $y > 8$. In set notation this would be $(-2, 0) \cup (8, \infty)$.

(a) $y(t)$ is decreasing where $\frac{dy}{dt}$ is negative. This happens when $y < -2$ and $0 < y < 8$. In set notation this would be $(-\infty, -2) \cup (0, 8)$.

(c) If $\frac{dy}{dt}$ is neither positive or negative, then it must be zero. This occurs at three points:

$$0.5y(2 + y)(y - 8) = 0$$
$$y = 0, -2, 8$$

These three values of y are called *equilibria*, since the constant functions $y = 0$, $y = -2$, and $y = 8$ are solutions of the differential equation. In other words, if your initial condition is any of these three values, then your solution will remain constant at that value for all t.

Answer to Question 6.
(a) In looking for the qualitative behavior of solutions, it's often best to look at the sign of $\frac{dy}{dt}$ and how it changes. Here, we see that both the r and y terms are positive, so the sign of $\frac{dy}{dt}$ is determined by the sign of $1 - \frac{y}{K}$.

This means that is $y > K$, then $\frac{dy}{dt} < 0$, or in other words that any solution above the line $y = K$ will be decreasing.
If $y = K$, then $\frac{dy}{dt} = 0$, which means that the constant function $y(t) = K$ is in fact a solution. This is called an *equilibrium* solution of the differential equation.
If $0 < y < K$, then $\frac{dy}{dt} > 0$, so any solution below the line $y = K$ will be increasing.

Putting this all together, this means that any solution starting below $y = K$ will increase asymptotically up towards the limit $y = K$ as $t \to \infty$. Similarly, any solution starting above $y = K$ will decrease asymptotically towards K, while the solution starting at K will remain there. So no matter what the initial population $y(0)$ is, it will approach the limit of K when $t \to \infty$. In population growth models, K functions here as the carrying capacity. Note that the constant r doesn't affect the limiting value of y, just the rate at which solutions approach the limit K.
If this is confusing, don't worry, we'll go over more problems like this in Section 2.5

(b) This part works similarly to part a. Now, we can rearrange the terms as:

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K} - \frac{E}{r}\right)y$$

Now we see that the sign of $\frac{dy}{dt}$ depends purely upon the sign of $1 - \frac{y}{K} - \frac{E}{r}$.
First, if $E \geq r$, then $1 - E/r < 0$, so this term is always negative for any $y > 0$. This means that $\frac{dy}{dt}$ will always be negative, and so all solutions $y(t)$ will eventually decrease to the limit of 0 as $t \to \infty$, i.e. the population will go extinct.

However, if $E < r$ (i.e. if there is less predation), then we calculate that $\frac{dy}{dt} = 0$ when

$$1 - \frac{y}{K} - \frac{E}{r} = 0$$

which we can solve for y as:

$$y = K\left(1 - \frac{E}{r}\right)$$

This quantity is the new carrying capacity: Any population y starting at this equilbrium value will stay there for all times t since $\frac{dy}{dt} = 0$
We can further see that if $y > K(1 - E/r)$, then $\frac{dy}{dt} < 0$, so all solutions starting above this value will decrease asymptotically to this new carrying capacity. Similarly, all solutions starting below $y = K(1 - E/r)$, will increase asymptotically up to this new carrying capacity.