Math 2930 Worksheet

Higher Order and Euler Equations

Week 9

October 19th, 2017

Question 1. (*) Find the general solution of the 4th order differential equation

$$y^{(4)} + 2y'' + y = 0$$

Question 2. (*) Consider the following two-point boundary value problem for $y(x)$:

$$\begin{aligned} y'' + \frac{\pi^2}{L^2}y &= p \\ y'(0) &= 0 \\ y(L) &= 0 \end{aligned}$$

where p is a given constant.

(a) Solve the boundary value problem.

(b) Neatly sketch the solution on $0 \le x \le L$. On your sketch label the y value at each of the end points.

Question 3. (*) Find the general solution of

$$x^2 \frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = x^5$$

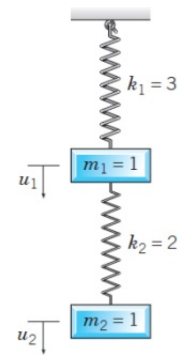
Question 4. Suppose there are two masses m_1 and m_2. m_1 is suspended by a spring hanging from the ceiling, and m_2 is suspended by another spring hanging from m_2 as in the picture below. Their positions u_1 and u_2 satisfy the coupled system of equations:

$$u_1'' + 5u_1 = 2u_2, \qquad u_2'' + 2u_2 = 2u_1 \tag{1}$$

(a) Solve the first equation of (1) for u_2 and substitute into the second equation, thereby obtaining the following fourth-order equation for u_1:

$$u_1^{(4)} + 7u_1'' + 6u_1 = 0 \tag{2}$$

Find the general solution of equation (2)



(b) Suppose that the initial conditions are:

$$u_1(0) = 1, \qquad u_1'(0) = 0, \qquad u_2(0) = 2, \qquad u_2'(0) = 0$$

Use these initial conditions and the first equation of (1) to obtain values for $u_1''(0)$ and $u_1'''(0)$.

(c) Show that the solution of Eq. (2) that satisfies the initial conditions you found in part **(b)** is

$$u_1(t) = \cos(t)$$

Question 5. Consider a horizontal metal beam of length L subject to a vertical load $f(x)$ per unit length. The resulting vertical displacement in the beam $y(x)$ satisfies a differential equation of the form

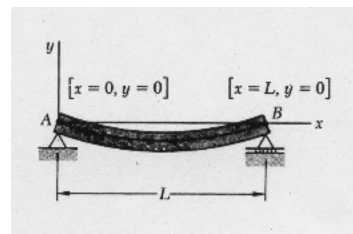
$$A\frac{d^4y}{dx^4} = f(x)$$

where A is a constant related to Young's modulus and the moment of inertia of the beam. (See picture below).
Suppose that $f(x)$ is a constant k:

$$A\frac{d^4y}{dx^4} = k$$

For each of the boundary conditions given below, solve for the displacement $y(x)$:
(a) Simply supported at both ends:

$$y(0) = y''(0) = y(L) = y''(L) = 0$$

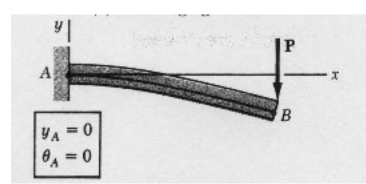


(b) Clamped at both ends:

$$y(0) = y'(0) = y(L) = y'(L) = 0$$

(c) Clamped at $x = 0$, free at $x = L$:

$$y(0) = y'(0) = y''(L) = y'''(L) = 0$$



Answer to Question 1.
(a)
Finding the roots of the characteristic polynomial,

$$\begin{aligned} r^4 + 2r^2 + 1 &= 0 \\ (r^2+1)(r^2+1) &= 0 \\ r = \pm i, \quad &\pm i \text{ (repeated)} \end{aligned}$$

The corresponding general solution is then

$$\boxed{y = c_1 \cos(t) + c_2 \sin(t) + c_3 t \cos(t) + c_4 t \sin(t)}$$

Answer to Question 2. (a) First, we find the solution of the homogenous equation:

$$y'' + \frac{\pi^2}{L^2} y = 0$$

Finding the roots of the characteristic polynomial,

$$\begin{aligned} r^2 + \frac{\pi^2}{L^2} &= 0 \\ r^2 &= -\frac{\pi^2}{L^2} \\ r &= \pm\sqrt{-\frac{\pi^2}{L^2}} = \pm\frac{\pi}{L} i \end{aligned}$$

So the homogenous solution is:

$$y_h(x) = c_1 \cos\left(\frac{\pi}{L}x\right) + c_1 \sin\left(\frac{\pi}{L}x\right)$$

Now for the particular solution. For the method of undetermined coefficients, we guess something in the form of the right hand side. Since the right hand side is just a constant, we guess a particular of the form

$$Y(x) = A$$

plugging this into the equation and solving for A,

$$\begin{aligned} 0 + \frac{\pi^2}{L^2} A &= p \\ A &= \frac{pL^2}{\pi^2} \end{aligned}$$

So the general solution is

$$y(x) = y_h(x) + Y(x) = c_1 \cos\left(\frac{\pi}{L}x\right) + c_1 \sin\left(\frac{\pi}{L}x\right) + \frac{pL^2}{\pi^2}$$

Now that we have the general solution, we plug in the boundary values to try and find c_1 and c_2. (*Note*: a common mistake is to try and find c_1 and c_2 before finding the particular solution, but this doesn't work. You need to find the particular solution first, and then find c_1 and c_2).

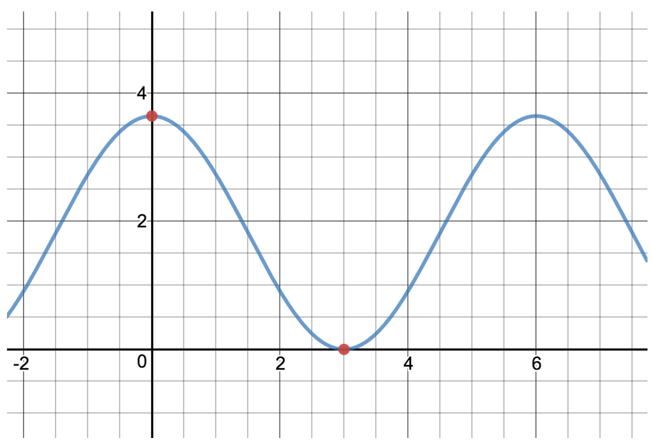
So plugging in the boundary values,

$$\begin{aligned}
y'(0) &= -\frac{\pi}{L}c_1 \sin(0) + \frac{\pi}{L}c_2 \cos(0) = 0 \\
y'(0) &= \frac{\pi}{L}c_2 = 0 \\
c_2 &= 0 \\
y(L) &= c_1 \cos\left(\frac{\pi}{L}L\right) + c_1 \sin\left(\frac{\pi}{L}L\right) + \frac{pL^2}{\pi^2} = 0 \\
y(L) &= c_1 \cos(\pi) + c_2 \sin(\pi) + \frac{pL^2}{\pi^2} = 0 \\
y(L) &= -c_1 + \frac{pL^2}{\pi^2} = 0 \\
c_1 &= \frac{pL^2}{\pi^2}
\end{aligned}$$

So the solution to this boundary value problem is

$$\boxed{y(x) = \frac{pL^2}{\pi^2}\left[\cos\left(\frac{\pi x}{L}\right) + 1\right]}$$

(b) A graph of the solution is here:



The endpoints are denoted by the red points in the circle above. The first is at $(0, 2pL^2/\pi^2)$ and the second is at $(L, 0)$.

Answer to Question 3. This equation is what is known as an *Euler equation*, and since the right-hand side is nonzero, this is a non-homogenous Euler equation.
So we will have to find both a homogenous and particular solution of the equation.
For the homogenous solution, we are solving:

$$x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = 0$$

Because this is an Euler equation, we guess a solution of the form $y = x^r$. Calculating derivatives,

$$\begin{aligned} y &= x^r \\ \frac{dy}{dx} &= rx^{r-1} \\ \frac{d^2y}{dx^2} &= r(r-1)x^{r-2} \end{aligned}$$

Plugging this into the original equation, and solving for r,

$$\begin{aligned} x^2 r(r-1)x^{r-2} + 4xrx^{r-1} + 2x^r &= 0 \\ r(r-1)x^r + 4rx^r + 2x^r &= 0 \\ \left[r(r-1) + 4r + 2\right]x^r &= 0 \\ r(r-1) + 4r + 2 &= 0 \\ r^2 + 3r + 2 &= 0 \\ (r+1)(r+2) &= 0 \\ r &= -1, \quad -2 \end{aligned}$$

So the homogenous solution is

$$y_h(x) = c_1 x^{-1} + c_2 x^{-2}$$

For the particular solution, we guess something in the form of the right hand side, so we'll guess

$$Y(x) = Ax^5$$

(*Note*: in the case of constant coefficients, we would usually need to add on a bunch of lower order terms, *i.e* $Bx^4 + Cx^3 + ...$ But for Euler equations, this is unnecessary, since we always get the same power of x back.)
The derivatives are

$$\begin{aligned} Y(x) &= Ax^5 \\ Y'(x) &= 5Ax^4 \\ Y''(x) &= 20Ax^3 \end{aligned}$$

Plugging this in,

$$\begin{aligned} x^2Y'' + 4xY' + 2Y &= x^5 \\ x^2(20Ax^3) + 4x(5Ax^4) + 2(Ax^5) &= x^5 \\ \left[20A + 20A + 2\right]x^5 &= x^5 \\ 42A &= 1 \\ A &= \frac{1}{42} \end{aligned}$$

So the particular solution is

$$Y(x) = \frac{1}{42}x^5$$

and the corresponding general solution is

$$\boxed{y(x) = y_h(x) + Y(x) = c_1\frac{1}{x} + c_2\frac{1}{x^2} + \frac{1}{42}x^5}$$

Answer to Question 4. (a)
Solving the first equation for u_2,

$$u_2 = \frac{u_1'' + 5u_1}{2}$$

then plugging this into the second equation,

$$\begin{aligned}
\left(\frac{u_1'' + 5u_1}{2}\right)'' + 2\left(\frac{u_1'' + 5u_1}{2}\right) &= 2u_1 \\
\frac{1}{2}u^{(4)} + \frac{5}{2}u_1'' + u_1'' + 5u_1 &= 2u_1 \\
u^{(4)} + (5+2)u_1'' + 10u_1 &= 4u_1 \\
u^{(4)} + 7u_1'' + 6u_1 &= 0
\end{aligned}$$

The general solution of this equation can be found by using the ansatz $u = e^{rt}$, giving us the characteristic polynomial:

$$r^{(4)} + 7r^2 + 6 = 0$$

To then find the roots, we factor as:

$$\begin{aligned}
(r^2+1)(r^2+6) &= 0 \\
r = \pm i, &\qquad \pm\sqrt{6}i
\end{aligned}$$

giving us a general solution of:

$$\boxed{u_1(t) = C_1\cos(t) + C_2\sin(t) + C_3\cos(\sqrt{6}t) + C_4\sin(\sqrt{6}t)}$$

(b) We'll have to translate the initial conditions on u_2 into initial conditions on u_1.
We can do that with the original equations relating u_1 and u_2 as follows:

$$\begin{aligned}
u_1''(t) + 5u_1(t) &= 2u_2(t) \\
u_1''(0) + 5u_1(0) &= 2u_2(0) \\
u_1''(0) + 5(1) &= 2(2) \\
u_1''(0) &= -1
\end{aligned}$$

and

$$\begin{aligned}
u_1''(t) + 5u_1(t) &= 2u_2(t) \\
u_1'''(t) + 5u_1'(t) &= 2u_2'(t) \\
u_1'''(0) + 5u_1'(0) &= 2u_2'(0) \\
u_1'''(0) + 5(0) &= 2(0) \\
u_1'''(0) &= 0
\end{aligned}$$

(c) So we have the general solution

$$u_1(t) = C_1 \cos(t) + C_2 \sin(t) + C_3 \cos(\sqrt{6}t) + C_4 \sin(\sqrt{6}t)$$

with initial conditions:

$$u_1(0) = 1, \qquad u_1'(0) = 0, \qquad u_1''(0) = -1, \qquad u_1'''(0) = 0$$

Plugging these in, we get the following equations:

$$\begin{aligned}
u_1(0) &= C_1 + 0 + C_3 + 0 = C_1 + C_3 = 1 \\
u_1'(0) &= 0 + C_2 + 0 + \sqrt{6}C_4 = C_2 + \sqrt{6}C_4 = 0 \\
u_1''(0) &= -C_1 + 0 - 6C_3 + 0 = -C_1 - 6C_3 = -1 \\
u_1'''(0) &= 0 - C_2 + 0 - 6\sqrt{6}C_4 = -C_2 - 6\sqrt{6}C_4 = 0
\end{aligned}$$

The second and fourth equations give us that $C_2 = C_4 = 0$.
The first and third equations are then:

$$\begin{aligned}
C_1 + C_3 &= 1 \\
-C_1 - 6C_3 &= -1
\end{aligned}$$

Adding the first equation to the second gives

$$\begin{aligned}
-5C_3 &= 0 \\
C_3 &= 0
\end{aligned}$$

which results in $C_1 = 1$. Overall, the constants are:

$$C_1 = 1, \qquad C_2 = 0, \qquad C_3 = 0, \qquad C_4 = 0$$

Therefore the specific solution to this IVP is:

$$\boxed{u_1(t) = \cos(t)}$$

(d) To find $u_2(t)$, we plug it into our original equation, and get:

$$\begin{aligned}
u_1''(t) + 5u_1(t) &= 2u_2(t) \\
(\cos(t))'' + 5\cos(t) &= 2u_2(t) \\
-\cos(t) + 5\cos(t) &= 2u_2(t)
\end{aligned}$$

$$\boxed{u_2(t) = 2\cos(t)}$$

Answer to Question 5. Here we have a non-homogenous 4th order equation:

$$Ay^{(4)} = k$$

We could solve this by using the characteristic polynomial, then find a particular solution, etc, but there's a much easier way: just integrate both sides four times.

$$\begin{aligned}
y^{(4)}(x) &= \frac{k}{A} \\
y'''(x) &= \frac{k}{A}x + c_1 \\
y''(x) &= \frac{k}{2A}x^2 + c_1x + c_2 \\
y'(x) &= \frac{k}{6A}x^3 + \frac{c_1}{2}x^2 + c_2x + c_3 \\
y(x) &= \frac{k}{24A}x^4 + \frac{c_1}{6}x^3 + \frac{c_2}{2}x^2 + c_3x + c_4
\end{aligned}$$

Of course this wouldn't work for most equations, since at some point we would get a term of $\int y(x)dx$, which we can't figure out since we don't know what $y(x)$ is. But in this case, since we only have the one term, it does work out.
(*Note*: often we would just use c_1 instead of $c_1/6$, but writing it this way means I can go ahead and use the derivatives of y I wrote up above for applying the boundary conditions. If I was changing what c_1, c_2, etc meant from line to line then I wouldn't be able to do that.)
(a) Plugging in the boundary conditions:

$$y(0) = y''(0) = y(L) = y''(L) = 0$$

we get the following system of 4 equations for (c_1, c_2, c_3, c_4):

$$\begin{aligned}
y(0) &= c_4 && = 0 \\
y''(0) &= c_2 && = 0 \\
y(L) &= \frac{k}{24A}L^4 + \frac{c_1}{6}L^3 + \frac{c_2}{2}L^2 + c_3L + c_4 && = 0 \\
y''(L) &= \frac{k}{2A}L^2 + c_1L + c_2 && = 0
\end{aligned}$$

Putting $c_2 = c_4 = 0$ into the last two equations gives

$$\begin{aligned}
\frac{k}{24A}L^4 + \frac{c_1}{6}L^3 + c_3L &= 0 \\
\frac{k}{2A}L^2 + c_1L &= 0
\end{aligned}$$

Solving the second equation for c_1,

$$c_1 = -\frac{kL}{2A}$$

plugging that in to solve for c_3,

$$\begin{aligned}\frac{k}{24A}L^4 + -\frac{kL}{12A}L^3 + c_3L = 0 \\ c_3L = \frac{kL^4}{24A} \\ c_3 = \frac{kL^3}{24A}\end{aligned}$$

so now that we have c_1, c_2, c_3, c_4, we plug these in to get the solution:

$$y(x) = \frac{k}{24A}x^4 + \frac{-kL}{12A}x^3 + \frac{kL^3}{24A}x$$

which we can rewrite more simply as:

$$\boxed{y(x) = \frac{k}{24A}\left[x^4 - 2Lx^3 + L^3x\right]}$$

(b) Plugging in the boundary conditions:

$$y(0) = y'(0) = y(L) = y'(L) = 0$$

we get the following system of 4 equations for (c_1, c_2, c_3, c_4):

$$\begin{aligned}y(0) &= c_4 &= 0 \\ y'(0) &= c_3 &= 0 \\ y(L) &= \frac{k}{24A}L^4 + \frac{c_1}{6}L^3 + \frac{c_2}{2}L^2 + c_3L + c_4 &= 0 \\ y'(L) &= \frac{k}{6A}L^3 + \frac{c_1}{2}L^2 + c_2L + c_3\end{aligned}$$

Putting $c_3 = c_4 = 0$ into the last two equations gives

$$\begin{aligned}\frac{k}{24A}L^4 + \frac{c_1}{6}L^3 + \frac{c_2}{2}L^2 = 0 \\ \frac{k}{6A}L^3 + \frac{c_1}{2}L^2 + c_2L = 0\end{aligned}$$

this is a linear system of 2 equations in 2 unknowns, its solution is:

$$c_1 = \frac{-2kL}{24}, \quad c_2 = \frac{kL^2}{24}$$

putting these back into our original formula, the solution is:

$$y(x) = \frac{k}{24A}x^4 + \frac{-2kL}{24A}x^3 + \frac{kL^2}{24A}x^2$$

which we can rewrite more simply as:

$$\boxed{y(x) = \frac{k}{24A}\left[x^4 - 2Lx^3 + L^2x^2\right]}$$

(c) Plugging in the boundary conditions:

$$y(0) = y'(0) = y''(L) = y'''(L) = 0$$

we get the following system of 4 equations for (c_1, c_2, c_3, c_4):

$$\begin{aligned} y(0) &= c_4 && = 0 \\ y'(0) &= c_3 && = 0 \\ y''(L) &= \frac{k}{2A}L^2 + c_1L + c_2 && = 0 \\ y'''(L) &= \frac{k}{A}L + c_1 && = 0 \end{aligned}$$

Solving the last equation for c_1,

$$c_1 = -\frac{kL}{A}$$

Plugging it in to find c_2,

$$\begin{aligned} \frac{kL^2}{2A} - \frac{kL^2}{A} + c_2 &= 0 \\ c_2 &= \frac{kL^2}{2A} \end{aligned}$$

plugging in c_1, c_2, c_3, and c_4, we get a solution of

$$y(x) = \frac{k}{24A}x^4 + \frac{-kL}{6A}x^3 + \frac{kL^2}{4A}x^2$$

which can be rewritten more simply as

$$\boxed{\frac{k}{24A}\left[x^4 - 4Lx^3 + 6L^2x^2\right]}$$