Math 2930 Worksheet

Forced Vibrations, Higher-Order ODEs

Week 8

October 12th, 2017

Question 1.
An undamped forced oscillator is described by the equation:

$$u'' + \lambda u' + u = F_0 \sin(\omega t), \qquad y(0) = a, \qquad y'(0) = b$$

where $\lambda > 0$.

(a) Find the steady state solution of this equation.

(b) Should the steady state solution depend on the initial conditions? Why or why not?

(c) For $\lambda << 1$, what value of ω maximizes the amplitude of the steady state solution?

(d) What is the amplitude of the steady state solution in the limit as $\omega \to 0$?

Question 2. The goal of this question is to walk you through how to solve an oscillator with a piecewise forcing function, such as:

$$u'' + u = F(t) = \begin{cases} F_0 t, & 0 \leq t < \pi \\ F_0(2\pi - t), & \pi < t \leq 2\pi \\ 0, & 2\pi < t \end{cases}$$

with initial conditions

$$u(0) = 0, \qquad u'(0) = 0$$

(a) First, solve on the interval $[0, \pi]$. Here, find the solution u of:

$$u'' + u = F_0 t, \qquad u(0) = 0, \qquad u'(0) = 0$$

(b) Plug $t = \pi$ into your answer from **(a)** to show that

$$u(\pi) = F_0 \pi, \qquad u'(\pi) = 2F_0$$

(c) Now solve on the interval $[\pi, 2\pi]$. That means find the solution u of:

$$u'' + u = F_0(2\pi - t), \qquad u(\pi) = F_0\pi, \qquad u'(\pi) = 2F_0$$

(d) Plug $t = 2\pi$ into your answer from **(c)** to show that

$$u(2\pi) = 0, \qquad u'(2\pi) = -4F_0$$

(e) Now solve on the interval $[2\pi, \infty)$. That means find the solution u of:

$$u'' + u = 0, \qquad u(2\pi) = 0, \qquad u'(2\pi) = -4F_0$$

(f)

Now we can string our solutions together to get the solution to the full problem. Write:

$$u(t) = \begin{cases} u_1(t), & 0 \leq t < \pi \\ u_2(t), & \pi < t \leq 2\pi \\ u_3(t), & 2\pi < t \end{cases}$$

where u_1 is your solution to part **(a)** , u_2 is your solution to part **(c)** , and u_3 is your solution to part **(e)** . This solves the original equation with the piecewise forcing function.

Suppose there are two masses m_1 and m_2. m_1 is suspended by a spring hanging from the ceiling, and m_2 is suspended by another spring hanging from m_2. Their positions u_1 and u_2 satisfy the coupled system of equations:

$$u_1'' + 5u_1 = 2u_2, \qquad u_2'' + 2u_2 = 2u_1 \tag{1}$$

(a) Solve the first equation of (1) for u_2 and substitute into the second equation, thereby obtaining the following fourth-order equation for u_1:

$$u_1^{(4)} + 7u_1'' + 6u_1 = 0 \tag{2}$$

Find the general solution of equation (2)

(b) Suppose that the initial conditions are:

$$u_1(0) = 1, \qquad u_1'(0) = 0, \qquad u_2(0) = 2, \qquad u_2'(0) = 0$$

Use these initial conditions and the first equation of (1) to obtain values for $u_1''(0)$ and $u_1'''(0)$.

(c) Show that the solution of Equation (2) that satisfies the initial conditions you found in part **(b)** is

$$u_1(t) = \cos(t)$$

(d) Show that the corresponding solution u_2 is $u_2(t) = 2\cos(t)$.

Answer to Question 1. (a) To find the steady state solution, we want to find a particular solution to

$$u'' + \lambda u' + u = F_0 \sin(\omega t)$$

We will do this using the method of undetermined coefficients. This means we guess a particular solution of the form

$$U(t) = A\cos(\omega t) + B\sin(\omega t)$$

and then calculate its derivatives:

$$\begin{aligned} U'(t) &= -A\omega\sin(\omega t) + B\omega\cos(\omega t) \\ U''(t) &= -A\omega^2\cos(\omega t) - B\omega^2\sin(\omega t) \end{aligned}$$

Plugging this into the left hand side of the original equation,

$$\begin{aligned} U'' + \lambda U' + U &= F_0\sin(\omega t) \\ \left[-A\omega^2\cos(\omega t) - B\omega^2\sin(\omega t)\right] + \lambda\left[-A\omega\sin(\omega t) + B\omega\cos(\omega t)\right] + \left[A\cos(\omega t) + B\sin(\omega t)\right] &= F_0\sin(\omega t) \\ \left[-B\omega^2 - A\lambda\omega + B\right]\sin(\omega t) + \left[-A\omega^2 + B\lambda\omega + A\right]\cos(\omega t) &= F_0\sin(\omega t) \end{aligned}$$

Setting like terms equal, we get two equations for A and B:

$$\begin{aligned} -B\omega^2 - A\lambda\omega + B &= F_0 \\ -A\omega^2 + B\lambda\omega + A &= 0 \end{aligned}$$

Solving the second equation for B,

$$B = \frac{A(\omega^2 - 1)}{\lambda\omega}$$

Plugging this into the first equation,

$$\begin{aligned} A\left(\frac{-\omega^2(\omega^2-1) - \lambda^2\omega^2 + (\omega^2-1)}{\lambda\omega}\right) &= F_0 \\ A &= \frac{-F_0\lambda\omega}{(\omega^2-1)^2 + \lambda^2\omega^2} \\ B &= \frac{A(\omega^2-1)}{\lambda\omega} \\ B &= \frac{-F_0(\omega^2-1)}{(\omega^2-1)^2 + \lambda^2\omega^2} \end{aligned}$$

So the steady state solution is:

$$\boxed{U(t) = \frac{-F_0\lambda\omega}{(\omega^2-1)^2 + \lambda^2\omega^2}\cos(\omega t) + \frac{-F_0(\omega^2-1)}{(\omega^2-1)^2 + \lambda^2\omega^2}\sin(\omega t)}$$

(b) The steady state does not depend on the initial conditions, since the steady state solution is the particular solution to this non-homogenous problem. The initial conditions will only affect the coefficients of the homogenous solution, which is not part of the steady state solution.

(c) The amplitude (squared) of the steady state solution is:

$$\begin{aligned}
R^2 &= A^2 + B^2 \\
R^2 &= \left(\frac{-F_0\lambda\omega}{(\omega^2-1)^2+\lambda^2\omega^2}\right)^2 + \left(-\frac{F_0(\omega^2-1)}{(\omega^2-1)^2+\lambda^2\omega^2}\right)^2 \\
R^2 &= \frac{F_0^2\left[(\omega^2-1)^2+\lambda^2\omega^2\right]}{\left[(\omega^2-1)^2+\lambda^2\omega^2\right]^2} \\
R^2 &= \frac{F_0^2}{(\omega^2-1)^2+\lambda^2\omega^2} \\
R &= \frac{F_0}{\sqrt{(\omega^2-1)^2+\lambda^2\omega^2}}
\end{aligned}$$

For $\lambda << 1$, *i.e.* very small values of λ, the $\lambda^2\omega^2$ term becomes arbitrarily small (but nonzero), so this expression is maximized at $\omega = 1$.
Note that we're saying $\lambda << 1$ to mean that λ is close to but not *exactly* zero. If λ were exactly zero, then we would not have a damped equation, and the homogenous solution would no longer disappear as $t \to \infty$, so we could no longer look only at the particular solution.

(d) As $\omega \to 0$, we see from our formula above that

$$R = \frac{F_0}{\sqrt{(0-1)^2+0}} = F_0$$

Again, we have to say $\omega \to 0$ rather than $\omega = 0$ here. If ω were *exactly* zero, then we would have a free vibration problem, whose steady state solution is zero.

Answer to Question 2.
First off, in all of the subparts, the homogenous solution will always be

$$u_h(t) = C_1\cos(t) + C_2\sin(t)$$

it's only the particular solution that changes.

(a) For the particular solution, we will use the method of undetermined coefficients. We guess that

$$U = At$$

Then plugging it into the original equation,

$$U'' + U = 0 + At = F_0 t$$

So $A = F_0$ and our general solution to this problem is:

$$u(t) = u_h(t) + U(t) = C_1\cos(t) + C_2\sin(t) + F_0 t$$

Plugging in the initial values,

$$\begin{aligned}
u(0) &= C_1\cos(0) + C_2\sin(0) + F_0(0) = C_1 = 0 \\
u'(0) &= -C_1\sin(0) + C_2\cos(0) + F_0 = C_2 + F_0 = 0
\end{aligned}$$

So this gives $C_1 = 0$, and $C_2 = -F_0$, giving us a specific solution of:

$$\boxed{u(t) = -F_0 \sin(t) + F_0 t}$$

(b) Plugging in $t = \pi$, we get

$$\boxed{u(\pi) = -F_0 \sin(\pi) + F_0\pi = F_0\pi}$$

and

$$u'(t) = -F_0 \cos(t) + F_0$$
$$\boxed{u'(\pi) = -F_0 \cos(\pi) + F_0 = 2F_0}$$

(c) Here, for the particular solution, we guess a solution of the form

$$U(t) = At + B$$

Plugging this in,

$$\begin{aligned} U'' + U = 0 + (At + B) &= F_0(2\pi - t) \\ At + B &= -F_0 t + 2\pi F_0 \end{aligned}$$

from which we see $A = -F_0$ and $B = 2\pi F_0$. That means our general solution is:

$$u(t) = u_h(t) + U(t) = C_1 \cos(t) + C_2 \sin(t) + F_0(2\pi - t)$$

Plugging in the initial conditions,

$$\begin{aligned} u(\pi) &= C_1 \cos(\pi) + C_2 \sin(\pi) + F_0(2\pi - \pi) = C_1 + \pi F_0 = \pi F_0 \\ u'(\pi) &= -C_1 \sin(\pi) + C_2 \cos(\pi) - F_0 = -C_2 - F_0 = 2F_0 \end{aligned}$$

this means that $C_1 = 0$ and $C_2 = -3F_0$. So the specific solution to this IVP is:

$$\boxed{u(t) = u_h(t) + U(t) = -3F_0 \sin(t) + F_0(2\pi - t)}$$

(d) Plugging in $t = 2\pi$ into our solution above,

$$\boxed{u(2\pi) = -3F_0 \sin(2\pi) + F_0(2\pi - 2\pi) = 0}$$

and

$$u'(t) = -3F_0 \cos(t) - F_0$$
$$\boxed{u'(2\pi) = -3F_0 \cos(2\pi) - F_0 = -3F_0 - F_0 = -4F_0}$$

(e) For this part, we want to solve the IVP:

$$u'' + u = 0, \qquad u(2\pi) = 0, \qquad u'(2\pi) = -4F_0$$

This is a homogenous problem, and the general solution is:

$$u(t) = C_1 \cos(t) + C_2 \sin(t)$$

Plugging in the initial conditions,

$$\begin{aligned} u(2\pi) &= C_1 \cos(2\pi) + C_2 \sin(2\pi) = C_1 = 0 \\ u'(2\pi) &= -C_1 \sin(2\pi) + C_2 \cos(2\pi) = C_2 = -4F_0 \end{aligned}$$

So this means that the specific solution is:

$$\boxed{u(t) = -4F_0 \sin(t)}$$

Patching all of our solutions together, the solution to the piecewise forcing problem is:

$$\boxed{u(t) = \begin{cases} -F_0 \sin(t) + F_0 t, & 0 \le t < \pi \\ -3F_0 \sin(t) + F_0(2\pi - t), & \pi < t \le 2\pi \\ -4F_0 \sin(t), & 2\pi < t \end{cases}}$$

Answer to Question 3. (a)
Solving the first equation for u_2,

$$u_2 = \frac{u_1'' + 5u_1}{2}$$

then plugging this into the second equation,

$$\begin{aligned} \left(\frac{u_1'' + 5u_1}{2}\right)'' + 2\left(\frac{u_1'' + 5u_1}{2}\right) &= 2u_1 \\ \frac{1}{2}u^{(4)} + \frac{5}{2}u_1'' + u_1'' + 5u_1 &= 2u_1 \\ u^{(4)} + (5+2)u_1'' + 10u_1 &= 4u_1 \\ u^{(4)} + 7u_1'' + 6u_1 &= 0 \end{aligned}$$

The general solution of this equation can be found by using the ansatz $u = e^{rt}$, giving us the characteristic polynomial:

$$r^{(4)} + 7r^2 + 6 = 0$$

To then find the roots, we factor as:

$$\begin{aligned} (r^2 + 1)(r^2 + 6) &= 0 \\ r = \pm i, &\qquad \pm\sqrt{6}i \end{aligned}$$

giving us a general solution of:

$$\boxed{u_1(t) = C_1 \cos(t) + C_2 \sin(t) + C_3 \cos(\sqrt{6}t) + C_4 \sin(\sqrt{6}t)}$$

(b) We'll have to translate the initial conditions on u_2 into initial conditions on u_1.

We can do that with the original equations relating u_1 and u_2 as follows:

$$\begin{aligned}
u_1''(t) + 5u_1(t) &= 2u_2(t) \\
u_1''(0) + 5u_1(0) &= 2u_2(0) \\
u_1''(0) + 5(1) &= 2(2) \\
u_1''(0) &= -1
\end{aligned}$$

and

$$\begin{aligned}
u_1''(t) + 5u_1(t) &= 2u_2(t) \\
u_1'''(t) + 5u_1'(t) &= 2u_2'(t) \\
u_1'''(0) + 5u_1'(0) &= 2u_2'(0) \\
u_1'''(0) + 5(0) &= 2(0) \\
u_1'''(0) &= 0
\end{aligned}$$

(c) So we have the general solution

$$u_1(t) = C_1 \cos(t) + C_2 \sin(t) + C_3 \cos(\sqrt{6}t) + C_4 \sin(\sqrt{6}t)$$

with initial conditions:

$$u_1(0) = 1, \qquad u_1'(0) = 0, \qquad u_1''(0) = -1, \qquad u_1'''(0) = 0$$

Plugging these in, we get the following equations:

$$\begin{aligned}
u_1(0) &= C_1 + 0 + C_3 + 0 = C_1 + C_3 = 1 \\
u_1'(0) &= 0 + C_2 + 0 + \sqrt{6}C_4 = C_2 + \sqrt{6}C_4 = 0 \\
u_1''(0) &= -C_1 + 0 - 6C_3 + 0 = -C_1 - 6C_3 = -1 \\
u_1'''(0) &= 0 - C_2 + 0 - 6\sqrt{6}C_4 = -C_2 - 6\sqrt{6}C_4 = 0
\end{aligned}$$

The second and fourth equations give us that $C_2 = C_4 = 0$.
The first and third equations are then:

$$\begin{aligned}
C_1 + C_3 &= 1 \\
-C_1 - 6C_3 &= -1
\end{aligned}$$

Adding the first equation to the second gives

$$\begin{aligned}
-5C_3 &= 0 \\
C_3 &= 0
\end{aligned}$$

which results in $C_1 = 1$. Overall, the constants are:

$$C_1 = 1, \qquad C_2 = 0, \qquad C_3 = 0, \qquad C_4 = 0$$

Therefore the specific solution to this IVP is:

$$\boxed{u_1(t) = \cos(t)}$$

(d) To find $u_2(t)$, we plug it into our original equation, and get:

$$u_1''(t) + 5u_1(t) = 2u_2(t)$$
$$(\cos(t))'' + 5\cos(t) = 2u_2(t)$$
$$-\cos(t) + 5\cos(t) = 2u_2(t)$$

$$\boxed{u_2(t) = 2\cos(t)}$$