Math 2930 Worksheet

2nd-order equations

Week 7

October 5th, 2017

Question 1. A vibrating system satisfies the equation

$$u'' + \lambda u' + u = 0$$

Find the value of the damping coefficient λ for which the quasi-period of the damped motion is 50% greater than the period of the corresponding undamped motion.

(Hint: We would say that $e^{at}\cos(bt)$ has "quasi-period" $2\pi/b$ since this is the period of the cosine part, but the whole function is not strictly periodic)

Question 2. The position of a certain undamped spring-mass system satisfies the initial value problem:

$$u'' + 2u = 0, \qquad u(0) = 0, \qquad u'(0) = 2$$

(a) Find the solution of this initial value problem

For the rest of this question, I would like you to use a graphing calculator (if you have one) or an online graphing tool (I personally recommend `desmos.com`). Laptops/calculators are strongly preferred, but you may use a smartphone if you have nothing else suitable.

(b) Plot u versus t and u' versus t on the same axes. What do the graphs of u and u' look like? What can you say about how the graphs of u and u' are related?

(c) Now plot u' versus u. By this I mean plot $u(t)$ and $u'(t)$ parametrically, with t as the parameter. This plot is known as a *phase plot*, and the uu' plane is called the *phase plane*.
(On Desmos, parametric plots should be formatted as $(x(t), y(t))$)
What does a periodic solution $u(t)$ look like in the phase plane?
What is the direction of motion on the phase plot as t increases?

(d) Repeat part **(c)** , but now with:

$$u'' + 0.25u' + 2u = 0, \qquad u(0) = 0, \qquad u'(0) = 2$$

How has the phase plot changed?

Question 3. I went over today how if a, b, c are positive constants, then all solutions of

$$ay'' + by' + cy = 0$$

approach 0 as $t \to \infty$.

(a) Now, for

$$ay'' + by' + cy = d$$

where a, b, and c are positive constants, and d is a (not necessarily positive) constant, show that all solutions approach d/c as $t \to \infty$.

(b) What happens if $c = 0$?

(c) What happens if $b = 0$ and $c = 0$?

Question 4. It's actually possible to combine both reduction of order *and* variation of parameters at once to find the general solution of a non-homogenous second-order equation with only one part of the homogenous solution. This question is going to help guide you through this process.

(a) Show that $y_1(t) = t^{-1}$ is a solution of the corresponding homogenous equation for:

$$y'' + \frac{7}{t}y' + \frac{5}{t^2}y = \frac{1}{t}, \quad t > 0 \tag{1}$$

(b) Let $y(t) = y_1(t)v(t) = t^{-1}v(t)$, and show that y satisfies Equation (1) if v is a solution of

$$\frac{1}{t}v'' + \frac{5}{t^2}v' = \frac{1}{t} \tag{2}$$

(c) Equation (2) is first-order linear in v'. Solve (2) for v. Then multiply v by y_1 to get the general solution of (1). You should see that this method simultaneously finds both the second homogenous solution y_2 *and* a particular solution Y.

Answer to Question 1. First, let's solve the undamped system:

$$u'' + u = 0$$

The characteristic polynomial is $r^2 + 1 = 0$, resulting in a general solution of

$$u(t) = c_1 \cos(t) + c_2 \sin(t)$$

Since sine and cosine have period 2π, we now want to find λ for which the quasi-period is 3π. For the damped system

$$u'' + \lambda u' + u = 0$$

Solving the characteristic polynomial,

$$\begin{aligned} r^2 + \lambda r + 1 &= 0 \\ r &= \frac{-\lambda \pm \sqrt{\lambda^2 - 4}}{2} \\ r &= \frac{-\lambda}{2} \pm \frac{\sqrt{\lambda^2 - 4}}{2} \end{aligned}$$

In order for the motion to be quasi-periodic, we would need the roots r to be complex. This happens when $\lambda^2 - 4 < 0$, yielding

$$r = \frac{-\lambda}{2} \pm \frac{\sqrt{4 - \lambda^2}}{2} i$$

This corresponds to a general solution

$$u(t) = c_1 e^{-\lambda t/2} \cos\left(\frac{\sqrt{4 - \lambda^2}}{2} t\right) + c_2 e^{-\lambda t/2} \sin\left(\frac{\sqrt{4 - \lambda^2}}{2} t\right)$$

The quasi-period of this motion is then

$$T = 2\pi \left(\frac{2}{\sqrt{4 - \lambda^2}}\right) = \frac{4\pi}{\sqrt{4 - \lambda^2}}$$

So we want to find the value of λ for which

$$\frac{4\pi}{\sqrt{4 - \lambda^2}} = 3\pi$$

Solving,

$$\begin{aligned} \frac{4}{3} &= \sqrt{4 - \lambda^2} \\ \frac{16}{9} &= 4 - \lambda^2 \\ \frac{16}{9} - \frac{36}{9} &= -\lambda^2 \end{aligned}$$

$$\boxed{\lambda = \sqrt{\frac{20}{9}} = \frac{2\sqrt{5}}{3}}$$

Answer to Question 2.
(a) We want to find the solution of

$$u'' + 2u = 0, \qquad u(0) = 0, \qquad u'(0) = 2$$

Solving for the roots of the characteristic polynomial,

$$\begin{aligned} r^2 + 2 &= 0 \\ r^2 &= -2 \\ r &= \pm\sqrt{2}i \end{aligned}$$

The corresponding general solution is

$$u(t) = c_1 \cos(\sqrt{2}t) + c_2 \sin(\sqrt{2}t)$$

Plugging in the initial condition $u(0) = 0$,

$$\begin{aligned} u(0) = c_1 \cos(0) + c_2 \sin(0) &= 0 \\ c_1 &= 0 \end{aligned}$$

Then plugging in the initial condition $u'(0) = 2$,

$$\begin{aligned} u'(t) &= -\sqrt{2}c_1 \sin(\sqrt{2}t) + \sqrt{2}c_2 \cos(\sqrt{2}t) \\ u'(0) = -\sqrt{2}c_1 \sin(0) + \sqrt{2}c_2 \cos(0) &= 2 \\ \sqrt{2}c_2 &= 2 \\ c_2 &= \sqrt{2} \end{aligned}$$

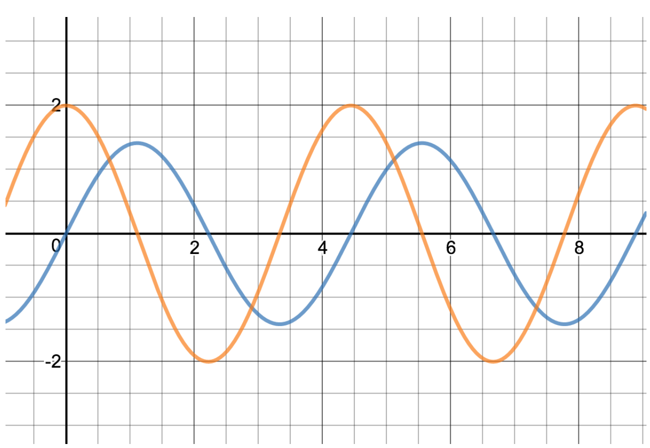
So the specific solution to this IVP is

$$\boxed{u(t) = \sqrt{2}\sin(\sqrt{2}t)}$$

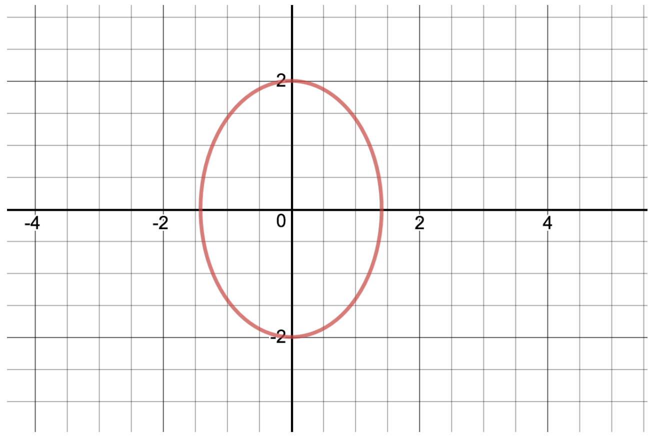
(b) The two functions we want to graph are

$$\begin{aligned} u(t) &= \sqrt{2}\sin(\sqrt{2}t) \\ u'(t) &= 2\cos(\sqrt{2}t) \end{aligned}$$

Graphed on the same axes, with u in blue and u' in orange



(c) If u and u' are plotted parametrically, the result is an ellipse:



Here, $u(t)$ and $u'(t)$ being periodic functions in t corresponds to this being a closed curve in the phase plane.
Increasing t corresponds to travelling clockwise around the ellipse.

(d) Now we want to repeat this with

$$u'' + 0.25u' + 2u = 0, \qquad u(0) = 0, \qquad u'(0) = 2$$

Solving for the roots of the characteristic equation,

$$r^2 + 0.25r + 2 = 0$$
$$r = \frac{-0.25 \pm \sqrt{0.25^2 - 8}}{2}$$
$$r = \frac{-1}{8} \pm \frac{1}{8}\sqrt{127}i$$

This has corresponding general solution

$$u(t) = c_1 e^{-t/8}\cos(\sqrt{127}t/8) + c_2 e^{-t/8}\sin(\sqrt{127}t/8)$$

Plugging in $u(0) = 0$ gets us that $c_1 = 0$, so

$$u(t) = c_2 e^{-t/8}\sin(\sqrt{127}t/8)$$

differentiating,

$$u'(t) = \frac{-c_2}{8}e^{-t/8}\sin(\sqrt{127}t/8) + \frac{\sqrt{127}c_2}{8}e^{-t/8}\cos(\sqrt{127}t/8)$$

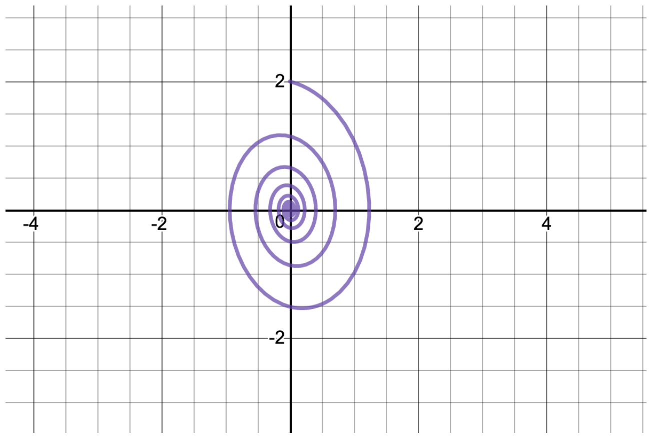
Plugging in $t = 0$,

$$u'(0) = \frac{\sqrt{127}c_2}{8} = 2$$
$$c_2 = \frac{16}{\sqrt{127}}$$

So the solution to this IVP is

$$\boxed{u(t) = \frac{16}{\sqrt{127}} e^{-t/8} \sin(\sqrt{127}t/8)}$$

Then the graph of u' versus u in the phase plane is this inward spiral:



Answer to Question 3.
(a) For

$$ay'' + by' + cy = d$$

Because this is a non-homogenous equation, we will need to use the method of undetermined coefficients. Our general solution will be $y(t) = y_h(t) + Y(t)$ where y_h is the homogenous solution, and Y is the particular solution to the non-homogenous problem.
With the method of undetermined coefficients, we "guess" a particular solution of the form

$$Y(t) = K$$

where K is a constant. Plugging this into the original equation,

$$\begin{aligned} aY'' + bY' + cY &= d \\ a(K)'' + b(K)' + c(K) &= d \\ 0 + 0 + cK &= d \\ K &= \frac{d}{c} \end{aligned}$$

So the general solution is

$$y(t) = y_h(t) + \frac{d}{c}$$

And since we know that the solution to the homogenous problem always approaches 0 as $t \to \infty$, we see that

$$\boxed{\lim_{t\to\infty} y(t) = \lim_{t\to\infty} y_h(t) + \lim_{t\to\infty} \frac{d}{c} = \frac{d}{c}}$$

So all solutions approach $\frac{d}{c}$ as $t \to \infty$.

(b) If $c = 0$, then our equation becomes

$$ay'' + by' = d$$

For the homogenous equation, our characteristic polynomial will be

$$ar^2 + br = r(ar + b)$$

resulting in a homogenous solution of

$$y_h(t) = c_1 + c_2 e^{-bt/a}$$

For the particular solution, we would normally guess $Y = K$ as in part **(a)** . But since constants are part of the homogenous solution, we will instead need to guess $Y = Kt$. Plugging this into the original equation,

$$\begin{aligned} aY'' + bY' &= d \\ a(0) + b(K) &= d \\ bK &= d \\ K &= \frac{d}{b} \end{aligned}$$

resulting in a particular solution of

$$Y(t) = \frac{dt}{b}$$

so the general solution is

$$\boxed{y(t) = y_h(t) + Y(t) = c_1 + c_2 e^{-bt/a} + \frac{dt}{b}}$$

So as $t \to \infty$, $y(t)$ will be dominated by the $\frac{dt}{b}$ term. $y(t)$ will approach either $+\infty$ or $-\infty$ depending upon the sign of d.

(c) If both $b = 0$ and $c = 0$, then our equation is

$$ay'' = d$$

This is technically a second-order constant coefficient non-homogenous equation. But it's simple enough that we can actually just integrate it twice rather than using the method of undetermined coefficients.

$$\begin{aligned} ay'' &= d \\ ay' &= dt + c_1 \\ ay &= \frac{dt^2}{2} + c_1 t + c_2 \\ y &= \frac{dt^2}{2a} + \frac{c_1 t}{a} + \frac{c_2}{a} \end{aligned}$$

So we see that as $t \to \infty$, the $\frac{dt^2}{2a}$ term will dominate, again going to $+\infty$ or $-\infty$ depending on the sign of d.

Answer to Question 4.
(a) First, we calculate the derivatives of y_1:

$$\begin{aligned} y_1(t) &= t^{-1} \\ y_1'(t) &= -t^{-2} \\ y_1''(t) &= 2t^{-3} \end{aligned}$$

Plugging these into the *homogenous* equation,

$$\begin{aligned} y'' + \frac{7}{t}y' + \frac{5}{t^2}y &= 0 \\ 2t^{-3} + 7(t^{-1})(-t^{-2}) + 5(t^{-2})(t^{-1}) &= 0 \\ (2-7+5)t^{-3} &= 0 \\ 0 &= 0 \end{aligned}$$

So we see that $y_1(t) = t^{-1}$ is in fact a solution to the homogenous equation.

(b) First we calculate the derivatives of y as follows

$$\begin{aligned} y &= t^{-1}v \\ y' &= -t^{-2}v + t^{-1}v' \\ y'' &= 2t^{-3}v - 2t^{-2}v' + t^{-1}v'' \end{aligned}$$

Plugging these into the *non-homogenous* equation,

$$\begin{aligned} y'' + 7t^{-1}y' + 5t^{-2}y &= t^{-1} \\ (2t^{-3}v - 2t^{-2}v' + t^{-1}v'') + 7(t^{-1})(-t^{-2}v + t^{-1}v') + 5(t^{-2})(t^{-1}v) &= t^{-1} \\ (2-7+5)t^{-3}v + (-2+7)t^{-2}v' + t^{-1}v'' &= t^{-1} \\ \frac{1}{t}v'' + \frac{5}{t^2}v' &= \frac{1}{t} \end{aligned}$$

(c) The previous equation is linear in v', so we can multiply by an integrating factor in order to solve. When written as above, the integrating factor is $\mu(t) = t^6$. So multiplying both sides by this factor,

$$\begin{aligned} (t^6)\frac{1}{t}v'' + (t^6)\frac{5}{t^2}v' &= (t^6)\frac{1}{t} \\ t^5v'' + 5t^4v' &= t^5 \\ (t^5v')' &= t^5 \\ t^5v' &= \int t^5 dt = \frac{t^6}{6} + c_2 \\ v' &= \frac{t}{6} + \frac{c_2}{t^5} \\ v &= \int \frac{t}{6} + \frac{c_2}{t^5} dt = \frac{t^2}{12} + \frac{c_2}{t^4} + c_1 \end{aligned}$$

Then multiplying by y_1 to get y:

$$y(t) = y_1(t)v(t)$$
$$y(t) = \frac{1}{t}\left(\frac{t^2}{12} + \frac{c_2}{t^4} + c_1\right)$$

giving a final answer of

$$\boxed{y(t) = \frac{t}{12} + \frac{c_2}{t^5} + \frac{c_1}{t}}$$

In this case we see that $y_1 = \frac{1}{t}$, $y_2 = \frac{1}{t^5}$, and $Y = \frac{t}{12}$.