Math 2930 Worksheet

2nd-order Homogenous Equations

Week 5

September 21st, 2017

Question 1.

(a) Solve the initial value problem

$$y'' + 2y' + 2y = 0, \qquad y(0) = 1, \qquad y'(0) = 0$$

(b) Graph the solution $y(t)$. How does y behave as $t \to \infty$?

Question 2.
The position u of a certain spring-mass system satisfies the initial value problem

$$\frac{3}{2}u'' + ku = 0, \qquad u(0) = 2, \qquad u'(0) = v$$

(a) Solve for the motion $u(t)$ in terms of k and v.

(b) The period and amplitude of the resulting motion are observed to be π and 3, respectively. Determine the values of k and v.
(Hint: $A\cos(t) + B\sin(t)$ has amplitude $\sqrt{A^2 + B^2}$)

Question 3.
The differential equation:

$$t^2\frac{d^2y}{dt^2} - 4t\frac{dy}{dt} + 6y = 0$$

is an example of what is known as an *Euler equation*.
(a) Upon first seeing equations like this, many students try to solve them in a way similar to solving equations with constant coefficients, which might look something like:

- "Guess" a solution of the form $y = e^{rt}$
- Write down the characteristic polynomial:

$$t^2r^2 - 4tr + 6 = 0$$

- Solve for r as a function of t using the quadratic formula
- Plug these two values of r back into $y = e^{rt}$, getting the two fundamental solutions as

$$y_1 = e^{r_1 t}, \qquad y_2 = e^{r_2 t}$$

But this actually doesn't work (it might be helpful to check this yourself).
Can you explain why this method that works for constant coefficients does not produce solutions here?

(b) Check that $y_1 = t^2$ is a solution of the original Euler equation.

(c) Using the solution given in part **(b)** , find the general solution using reduction of order.

(d) The disadvantage of using reduction of order is that it requires already knowing one solution to the original equation.
If you don't know a solution already, then Euler equations can be solved more generally using a change of variables. For the substitution $x = \ln(t)$, use the Chain Rule to show that:

$$\frac{dy}{dx} = t\frac{dy}{dt}, \qquad \frac{d^2y}{dx^2} = t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt}$$

(e) This change of variables reduces the Euler equation for $y(t)$ into a 2nd-order constant coefficient equation for $y(x)$:

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$

Solve this for $y(x)$.

(f) Plug $x = \ln(t)$ back in, finding the general solution $y(t)$. (i.e. undo the change of variables).
Hint: Your general solution should include $y_1 = t^2$ from part **(b)** .

Answer to Question 1. (a) This equation has constant coefficients, so the characteristic polynomial is:

$$r^2 + 2r + 1 = 0$$

We find the roots using the quadratic equation:

$$r = \frac{-2 \pm \sqrt{4 - 4(1)(2)}}{2} = \frac{-2 \pm \sqrt{-4}}{2} = -1 \pm i$$

So the general form of the solution is:

$$y(t) = C_1 e^{-t} \cos(t) + C_2 e^{-t} \sin(t)$$

Now we will need to use the initial conditions to find C_1 and C_2. We calculate the derivative of y:

$$y'(t) = \left[-C_1 e^{-t} \cos(t) - C_1 e^{-t} \sin(t)\right] + \left[-C_2 e^{-t} \sin(t) + C_2 e^{-t} \cos(t)\right]$$

Plugging in the initial conditions,

$$y(0) = C_1(1) + C_0(0) = C_1 = 1$$

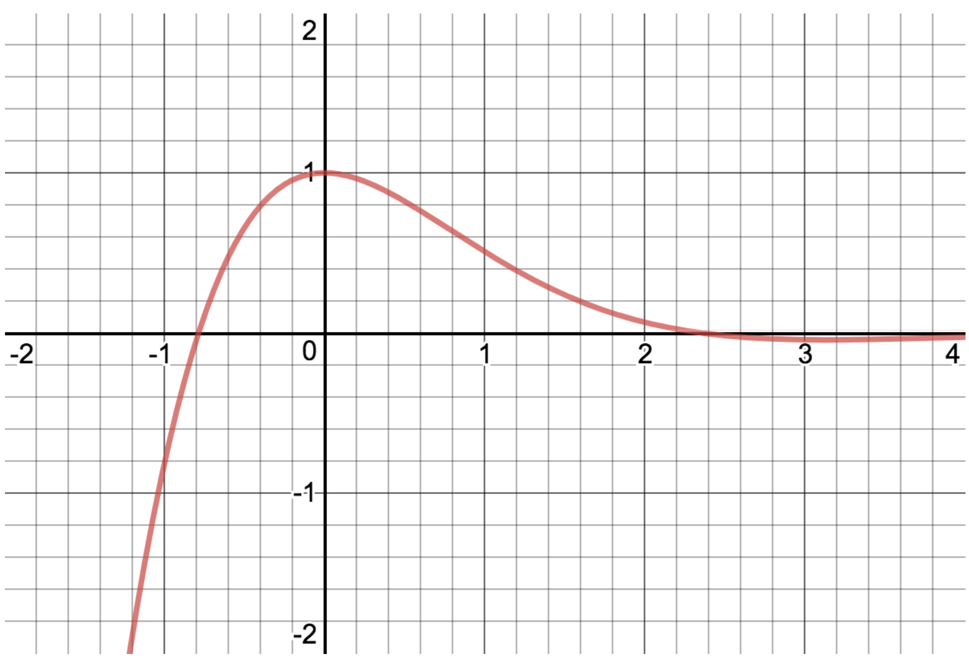
and

$$y'(0) = \left[-C_1 - 0\right] + \left[0 + C_2\right] = C_2 - C_1 = 0$$

So $C_1 = C_2 = 1$ are our two constants.
Therefore the solution to the initial value problem is:

$$y(t) = e^{-t} \cos(t) + e^{-t} \sin(t)$$

(b) We see that this function is a decaying oscillation that starts at $y = 1$ with a horizontal tangent line there, and has "quasi-period" 2π. It's graph will look like:



Answer to Question 2. (a) The characteristic polynomial is:

$$\frac{3}{2}r^2 + k = 0$$

Solving for r,

$$r^2 = \frac{-2k}{3}$$
$$r = \pm\sqrt{\frac{-2k}{3}} = \pm\sqrt{\frac{2k}{3}}i$$

So the general solution is:

$$u(t) = C_1 \cos\left(\sqrt{\frac{2k}{3}}t\right) + C_2 \sin\left(\sqrt{\frac{2k}{3}}t\right)$$

Now we use the initial conditions to solve for C_1 and C_2.

$$u(0) = C_1(1) + C_2(0) = C_1 = 2$$

and

$$u'(0) = -\sqrt{\frac{2k}{3}}C_1(0) + \sqrt{\frac{2k}{3}}C_2(1) = \sqrt{\frac{2k}{3}}C_2(1) = v$$

(Don't forget the Chain Rule in calculating u')

So $C_2 = v\sqrt{\frac{3}{2k}}$. Putting these back into $u(t)$ we get:

$$u(t) = 2\cos\left(\sqrt{\frac{2k}{3}}t\right) + v\sqrt{\frac{3}{2k}}\sin\left(\sqrt{\frac{2k}{3}}t\right)$$

(b) A full period is the time it takes for the quantity inside the sin and cos functions to change from 0 to 2π. So we can solve for the period T as follows:

$$2\pi = \sqrt{\frac{2k}{3}}T$$
$$T = \frac{2\pi\sqrt{3}}{\sqrt{2k}} = \frac{\pi\sqrt{6}}{\sqrt{k}}$$

Since the problem gives us that the period T is π,

$$\pi = \frac{\pi\sqrt{6}}{\sqrt{k}}$$
$$k = 6$$

Now to solve for v.

Our amplitude is given by:

$$3 = \sqrt{2^2 + \left(v\sqrt{\frac{3}{2k}}\right)^2}$$

So we will solve this equation for v. Squaring both sides,

$$9 = 4 + \frac{3v^2}{2k}$$

Since we already figured out that $k = 6$,

$$\begin{aligned} 5 &= \frac{v^2}{4} \\ v^2 &= \frac{5}{4} \\ v &= \pm\sqrt{\frac{5}{4}} \end{aligned}$$

Answer to Question 3. (a) In order to get this characteristic polynomial, we assumed that r was a constant, and not a function of t.
If we then solve that characteristic polynomial using the quadratic formula, we are then saying that r is a function of t, making the derivation of the characteristic equation we just solved incorrect. If we properly thought of r as a function of t, then we would also have derivatives of r in our "characteristic equation" from the chain rule, and we would generally be no better off than we started.

(b) Calculating derivatives,

$$\begin{aligned} y_1 &= t^2 \\ \frac{dy_1}{dt} &= 2t \\ \frac{d^2y_1}{dt^2} &= 2 \end{aligned}$$

Plugging these in,

$$t^2(2) - 4t(2t) + 6(t^2) = (2 - 8 + 6)t^2 = 0$$

So $y_1 = t^2$ is a solution of this Euler equation.

(c) For reduction of order, we will look for a solution of the form $y(t) = v(t)y_1(t)$. Taking derivatives,

$$\begin{aligned} y &= vy_1 \\ y' &= v'y_1 + vy_1' \\ y'' &= v''y_1 + 2v'y_1' + vy_1'' \end{aligned}$$

Plugging these into the original equation,

$$\begin{aligned} t^2y'' - 4ty' + 6y &= 0 \\ t^2\left[v''y_1 + 2v'y_1' + vy_1''\right] - 4t\left[v'y_1 + vy_1'\right] + 6\left[vy_1\right] &= 0 \end{aligned}$$

Rearranging according to the v terms, we get:

$$[t^2y_1]v'' + [2t^2y_1' - 4ty_1]v' + [t^2y_1'' - 4ty_1' + 6y_1]v = 0$$

Since we figured out in part **(a)** that y_1 is a solution of the original equation, this means that the quantity in front of the v term is actually just 0. So our equation for v becomes:

$$[t^2y_1]v'' + [2t^2y_1' - 4ty_1]v'$$

Then plugging in $y_1 = t^2$, we get

$$[t^4]v'' + [2t^2(2t) - 4t(t^2)]v' = t^4v'' = 0$$

So solving for v,

$$\begin{aligned} v'' &= \frac{0}{t^4} = 0 \\ v' &= \int 0dt = C_1 \\ v &= \int vdt = \int C_1dt = C_1t + C_2 \end{aligned}$$

So that means our general solution is:

$$\begin{aligned} &y(t) = v(t)y_1(t) = t^2[C_1t + C_2] \\ &y(t) = C_1t^3 + C_2t^2 \end{aligned}$$

Which we see does include our solution of t^2 from part **(b)** .

(d) If we have $x = \ln(t)$, then we calculate

$$\begin{aligned} \frac{dx}{dt} &= \frac{1}{t} \\ \frac{dt}{dx} &= t \end{aligned}$$

With this, we can use the Chain Rule to relate derivatives in x to derivatives in t:

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}t$$

Then for the second derivative,

$$\frac{d^2y}{dx}^2 = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dx}\left[\frac{dy}{dt}t\right] = \frac{d}{dt}\left[\frac{dy}{dt}t\right] \cdot \frac{dt}{dx}$$

Then using the product rule,

$$\frac{d^2y}{dx^2} = \left[\frac{d^2y}{dt^2}t + \frac{dy}{dt}\right] \cdot \frac{dt}{dx} = \left[\frac{d^2y}{dt^2}t + \frac{dy}{dt}\right]t = t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt}$$

(e) Solving the characteristic polynomial,

$$r^2 - 5r + 6 = 0$$
$$(r-2)(r-3) = 0$$
$$r = 2, \quad 3$$

So the general solution is:

$$y(x) = C_1 e^{2x} + C_2 e^{3x}$$

(f) Plugging $x = \ln(t)$ into our answer from **(e)** ,

$$y(t) = C_1 e^{2\ln(t)} + C_2 e^{3\ln(t)}$$
$$y(t) = C_1 t^2 + C_2 t^3$$

which matches our answer from part **(c)** , except this way we didn't need to be given $y_1 = t^2$ beforehand.