Math 2930 Worksheet

Exact Equations

Week 4

September 14, 2017

Question 1.
Separable equations are those that can be written in the form:

$$f(y)\frac{dy}{dx} = g(x)$$

When are these equations also *exact*? Explain your reasoning.

Question 2. (a) Find the value(s) of b for which the given equation is exact:

$$(xy^2 + bx^2y) + (x + y)x^2y' = 0$$

(b) Solve it for the value of b you found in part **(a)** .

Question 3. (a) Show that the equation below is *not* exact:

$$x^2y^3 + x(1 + y^2)y' = 0$$

(b) Show that it can be made exact by multiplying both sides of the equation by the integrating factor $\mu(x, y) = \frac{1}{xy^3}$.

(c) Now that the equation is exact, solve it.

Question 4. (a) Show that the equation below is *not* exact:

$$y + (2xy - e^{-2y})y' = 0$$

(b) It turns out that we can make this equation exact by using some sort of integrating factor μ (like we did in the previous question).

In order to find μ, we'll have to assume that it depends on either x only or on y only, but not both. If μ were to depend on x only, then we would want to get an ordinary differential equation for μ that would also only involve x only, and not y, so that we could solve for $\mu(x)$. And vice versa for μ depending on y only.

Which one of these approaches works for this equation and why?

(c) Solve the differential equation you found in part **(b)** for μ

(d) Solve the original equation in part **(a)** using the integrating factor μ you found in part **(c)**

(e) Can you extend your argument from part **(b)** to more general equations?

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0$$

i.e. what are the conditions on M and N such that an integrating factor $\mu = \mu(x)$ can be used to make the equation exact? When can an integrating factor $\mu = \mu(y)$ be used?

Answer to Question 1.
Rewriting this equation in the usual form, we have

$$g(x) - f(y)\frac{dy}{dx} = 0$$

So we have $M(x, y) = g(x)$ and $N(x, y) = -f(y)$.

In order for the equation to be exact, we need that $M_y = N_x$.
In this case, $M_y = \frac{\partial}{\partial y}[g(x)] = 0$ and $N_x = -\frac{\partial}{\partial x}[f(y)] = 0$ no matter what f and g are, so separable equations are always exact.

Answer to Question 2.
(a) This equation is exact when:

$$M_y = N_x$$

So for this problem, that becomes:

$$\frac{\partial}{\partial y}\left[xy^2 + bx^2y\right] = \frac{\partial}{\partial x}\left[(x+y)x^2\right]$$

Simplifying and solving for b,

$$2xy + bx^2 = 3x^2 + 2xy$$
$$bx^2 = 3x^2$$
$$b = 3$$

Therefore $b = 3$ is the only value of b for which the given equation is exact.

(b) For $b = 3$, we now want to find a function $\psi(x, y)$ such that:

$$\frac{\partial\psi}{\partial x} = M(x, y) = xy^2 + 3x^2y$$

and

$$\frac{\partial\psi}{\partial y} = N(x, y) = x^3 + x^2y$$

Integrating the first equation with respect to x (while holding y constant), we get:

$$\psi(x, y) = \frac{x^2y^2}{2} + x^3y + f(y)$$

for some function $f(y)$.
Differentiating with respect to y and setting it equal to N, we have

$$\frac{\partial\psi}{\partial y}x^2y + x^3 + f'(y) = N(x, y) = x^3 + x^2y$$

So clearly $f'(y) = 0$, which means that $f(y)$ is a constant.
Therefore the general solution is:

$$\psi(x, y) = \frac{x^2y^2}{2} + x^3y = C$$

Answer to Question 3. (a) For this equation:

$$M(x, y) = x^2y^3, \qquad N(x, y) = x(1 + y^2)$$

To check if it's exact, we calculate

$$\frac{\partial M}{\partial y} = 3x^2y^2$$

and

$$\frac{\partial N}{\partial x} = 1 + y^2$$

Since $M_y \neq N_x$, this equation is not exact.

(b) Multiplying everything by the integrating factor $\mu(x, y) = \frac{1}{xy^3}$,

$$M(x, y) = x$$

$$\frac{\partial M}{\partial y} = 0$$

and

$$N(x, y) = \frac{1}{y^3} + \frac{1}{y}$$

$$\frac{\partial N}{\partial x} = 0$$

Since $M_y = N_x$, the equation

$$x + \left(\frac{1}{y^3} + \frac{1}{y}\right) y' = 0$$

is exact (in fact it is actually separable).

(c) We want to find a function $\psi(x, y)$ with partial derivatives:

$$\frac{\partial \psi}{\partial x} = x, \qquad \frac{\partial \psi}{\partial y} = \frac{1}{y^3} + \frac{1}{y}$$

Integrating the first of those, we get

$$\psi(x, y) = \frac{1}{2}x^2 + f(y)$$

for some function $f(y)$. Integrating the second equation,

$$\psi(x, y) = \frac{-1}{2y^2} + \ln(y) + g(x)$$

Combining these two, we find that the solution is:

$$\psi(x, y) = \frac{1}{2}x^2 + \frac{-1}{2y^2} + \ln(y) = C$$

Answer to Question 4. (a) For this equation,

$$M(x, y) = y, \qquad N(x, y) = 2xy - e^{-2y}$$

To check if it is exact, we compute

$$\frac{\partial M}{\partial y} = 1$$

and

$$\frac{\partial N}{\partial x} = 2y$$

Since $M_y \neq N_x$, this equation is not exact.

(b) So we want to come up with an integrating factor μ so that if we multiply the entire equation by μ, then our equation is now exact.
There are two approaches we can try: either assuming that $\mu = \mu(x)$ only, or that $\mu = \mu(y)$ only. If we make these assumptions, it may be possible that we get an ODE we can solve to find μ.

So what happens if we assume $\mu = \mu(x)$ only? Our equation becomes

$$y\mu(x) + \mu(x)\left(2xy - e^{-2y}\right)\frac{dy}{dx} = 0$$

Now we check the condition for exactness:

$$\frac{\partial M}{\partial y} = \mu(x)$$

$$\frac{\partial N}{\partial x} = \mu'(x)\left(2xy - e^{-2y}\right) + \mu(x)2y$$

So this equation is exact if:

$$\mu(x) = \mu'(x)\left(2xy - e^{-2y}\right) + \mu(x)2y$$

Since this expression depends on both x and y, there is not a function $\mu(x)$ that will satisfy this equation.

Now, let's try an integrating factor of the form $\mu = \mu(y)$. Our equation becomes

$$y\mu(y) + \mu(y)\left(2xy - e^{-2y}\right)\frac{dy}{dx} = 0$$

Now we check the condition for exactness:

$$\frac{\partial M}{\partial y} = y\mu'(y) + \mu(y)$$

$$\frac{\partial N}{\partial x} = \mu(y)2y$$

So this equation is exact if:

$$y\mu'(y) + \mu(y) = \mu(y)2y$$

After rearranging,

$$\mu'(y) + \left(\frac{1}{y} - 2\right)\mu(y) = 0$$

Which we see is a first-order linear ODE in y. So if $\mu(y)$ is a solution of this ODE, it is an integrating factor for the original equation.

(c) This linear ODE can be solved as usual, using (another) integrating factor. This means we want to multiply both sides by some function η so that the left hand side looks like a product rule.

$$\eta\mu'(y) + \eta\left(\frac{1}{y} - 2\right)\mu(y) = 0$$

In order for this to be a product rule, we would need

$$\frac{d\eta}{dy} = \eta\left(\frac{1}{y} - 2\right)$$

This can be separated:

$$\int \frac{d\eta}{\eta} = \int \left(\frac{1}{y} - 2\right) dy$$

Solving for η,

$$\ln(\eta) = \ln(y) - 2y + C$$
$$\eta = Cye^{-2y}$$

Since the constant doesn't matter for an integrating factor, we'll just take $C = 1$. So plugging this back in, the ODE for $\mu(y)$ becomes:

$$ye^{-2y}\mu'(y) + \left(e^{-2y} - 2ye^{-2y}\right)\mu(y) = 0$$

The left hand side is now in the form of a product rule:

$$\left(ye^{-2y}\mu(y)\right)' = 0$$

Integrating both sides,

$$ye^{-2y}\mu(y) = C$$

Again, since μ is an integrating factor, the choice of constant C does not matter, so we will take $C = 1$ for simplicity. Then we find the integrating factor is:

$$\mu(y) = \frac{e^{2y}}{y}$$

(d) Multiplying both sides of the original equation by the integrating factor $\mu(y) = \frac{e^{2y}}{y}$,

$$e^{2y} + \left(2xe^{2y} - \frac{1}{y}\right)\frac{dy}{dx} = 0$$

We can check that the equation is now exact:

$$\frac{\partial M}{\partial y} = 2e^{2y}$$

$$\frac{\partial N}{\partial x} = 2e^{2y}$$

So we want to look for a solution of the form $\psi(x, y) = C$. ψ must have partial derivatives:

$$\frac{\partial \psi}{\partial x} = M(x, y) = e^{2y}$$

$$\frac{\partial \psi}{\partial y} = N(x, y) = 2xe^{2y} - \frac{1}{y}$$

Integrating the first equation with respect to x,

$$\psi(x, y) = xe^{2y} + f(y)$$

for some function $f(y)$.
Integrating the second equation with respect to y,

$$\psi(x, y) = xe^{2y} - \ln(y) + g(x)$$

for some function $g(y)$.
Putting these two together, we see that the final solution is

$$\psi(x, y) = xe^{2y} - \ln(y) = C$$

(e) In general, if we multiply by an integrating factor $\mu(x)$, we have

$$\mu(x)M(x, y) + \mu(x)N(x, y)\frac{dy}{dx} = 0$$

This equation is exact if:

$$\frac{\partial}{\partial y}\left[\mu(x)M(x, y)\right] = \frac{\partial}{\partial x}\left[\mu(x)N(x, y)\right]$$

$$\mu(x)M_y = \mu'(x)N + \mu(x)N_x$$

Rearranging,

$$\mu'(x) = \left(\frac{M_y - N_x}{N}\right)\mu(x)$$

In general, the $\frac{M_y - N_x}{N}$ term can depend on both x and y. In order to be able to solve for μ as a function of x, we then need for this term to depend only on x. In other words, an integrating factor of the form $\mu(x)$ can be found if:

$$\frac{M_y - N_x}{N} \quad \text{is a function of } x \text{ only}$$

We can repeat the same process with $\mu(y)$ instead of $\mu(x)$. Our equation becomes:

$$\mu(y)M(x, y) + \mu(y)N(x, y)\frac{dy}{dx} = 0$$

Checking for exactness,

$$\frac{\partial}{\partial y}\left[\mu(y)M(x, y)\right] = \frac{\partial}{\partial x}\left[\mu(y)N(x, y)\right]$$

$$\mu(y)M_y + \mu'(y)M = \mu(y)N_x$$

Rearranging,

$$\mu'(y) = \left(\frac{N_x - M_y}{M}\right)\mu(y)$$

So an integrating factor of the form $\mu(y)$ can be used if and only if:

$$\frac{N_x - M_y}{M} \quad \text{is a function of } y \text{ only}$$