Math 2930 Worksheet

Equilibria and Stability

Week 3

September 7, 2017

Question 1. (a) Let C be the temperature (in Fahrenheit) of a cup of coffee that is cooling off to room temperature.
Which of the following differential equations best models the situation and why?

Hint: What should the equilibrium (or equilibria) of this equation be? What about the stability?

$$\frac{dC}{dt} = 0.4(C - 70)$$

$$\frac{dC}{dt} = 0.4C\left(1 - \frac{C}{70}\right)$$

$$\frac{dC}{dt} = -0.4C + 28$$

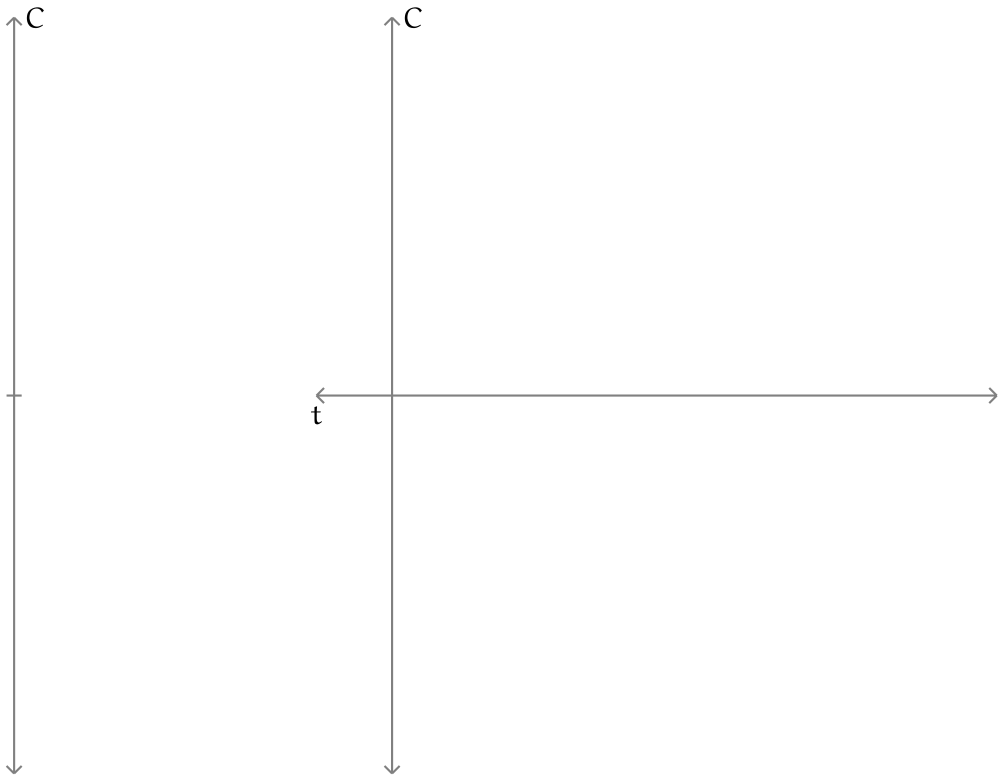
$$\frac{dC}{dt} = -0.4C$$

$$\frac{dC}{dt} = -0.4(C - 70)$$

$$\frac{dC}{dt} = -0.4(C - 28)$$

$$\frac{dC}{dt} = -0.4(C - 70)^2$$

(b) For the equation you picked in part **(a)**, first draw its phase line. Then sketch some solutions for different initial values in the C–t plane provided below. Be sure to label any equilibria and their stability.



(c) Consider a situation where there are two separate cups of coffee. At $t = 0$, one of the cups starts at a temperature of $C = 180$, while the other starts at a temperature of $C = 160$.

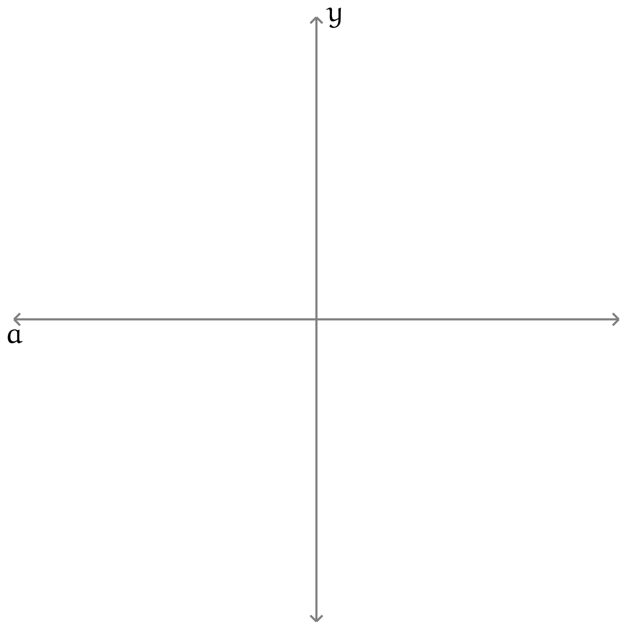
Will there ever be a time t at which the temperatures of the two cups are *exactly* the same? Why or why not?

Question 2. (a) Consider the autonomous ODE:

$$\frac{dy}{dt} = a - y^2$$

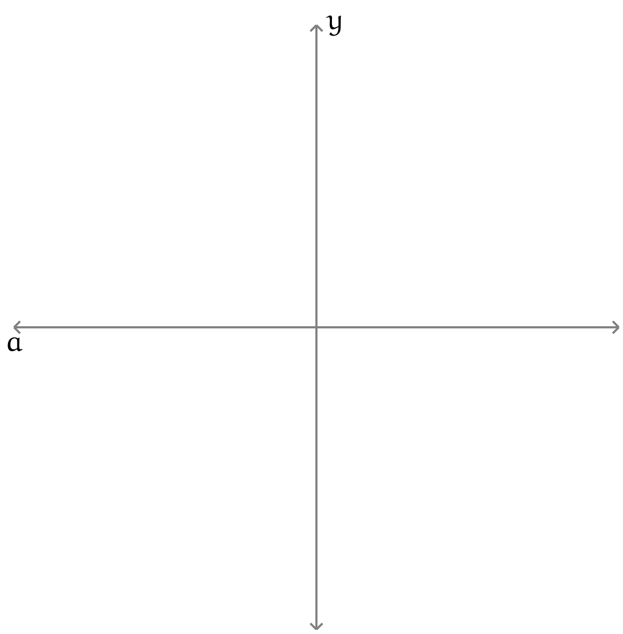
Here a is a *parameter* that does not depend on y or t.

The number of equilibria and their stability will depend on the value of a.
On the *a*–*y* axes below, plot the different equilibria y versus the value of the parameter a.
Label which parts of the graph correspond to stable and unstable equilibria.



(b) Repeat part **(a)**, but this time with:

$$\frac{dy}{dt} = ay - y^3$$



These diagrams in the *a*–*y* plane are known as *bifurcation diagrams*. Part **(a)** is a *saddle-node* bifurcation and part **(b)** is a *pitchfork* bifurcation.

Question 3. For each part below, create a continuous autonomous differential equation that has the stated properties (if possible). If it is not possible, provide a justification for why not.
(a) Exactly three equilibria, two unstable and one stable

(b) Exactly two equilibria, both stable

(c) Exactly two equilibria, one unstable and one semistable

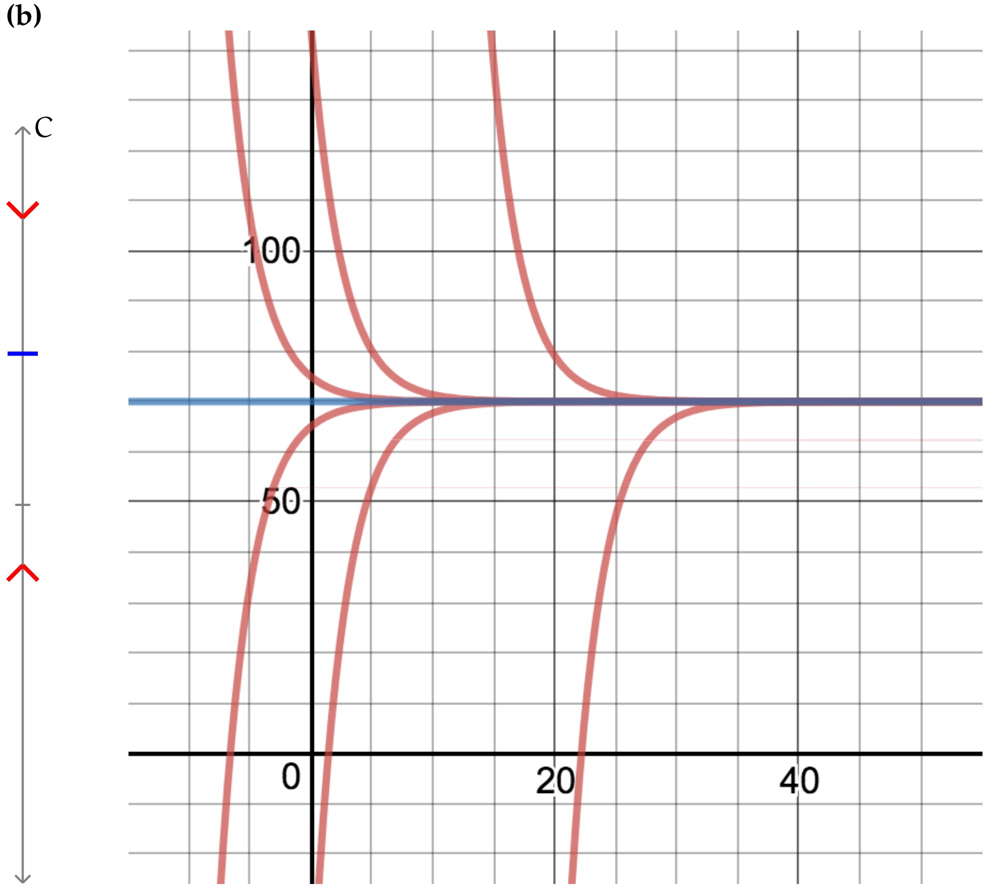
Question 4. For each part below, create a continuous autonomous differential equation that has the given functions as solutions:
(a) $y(t) = e^{2t+1}$ is a solution

(b) $y(t) = 1 - e^{-t}$ and $y(t) = 1 + e^{-t}$ are solutions

Answer to Question 1. (a) The temperature of the coffee should approach the room temperature of 70 Fahrenheit. So we want an equation which has only one equilbrium at $C = 70$, and for this equilibrium to be stable. This would mean that all solutions eventually approach $C = 70$ regardless of initial condition.

The only models that satisfy this criteria are $\frac{dC}{dt} = -0.4C + 28$ and $\frac{dC}{dt} = -0.4(C - 70)$, which are actually the same. All the other equations have equilibria at points other than $C = 70$, or the equilbrium is either stable or unstable. This answer is in fact exactly what would be given by Newton's Law of Cooling with constant 0.4 (see HW problem 1.1.19).

(b)



(c) In this case, the two cups of coffee will both asymptotically approach the equilibrium temperature of 70 degrees, but there is no point in time at which they will have *exactly* the same temperature.
One way of thinking about this is that being the same temperature would mean that these solutions cross in the $C - t$ plane, which should never happen.

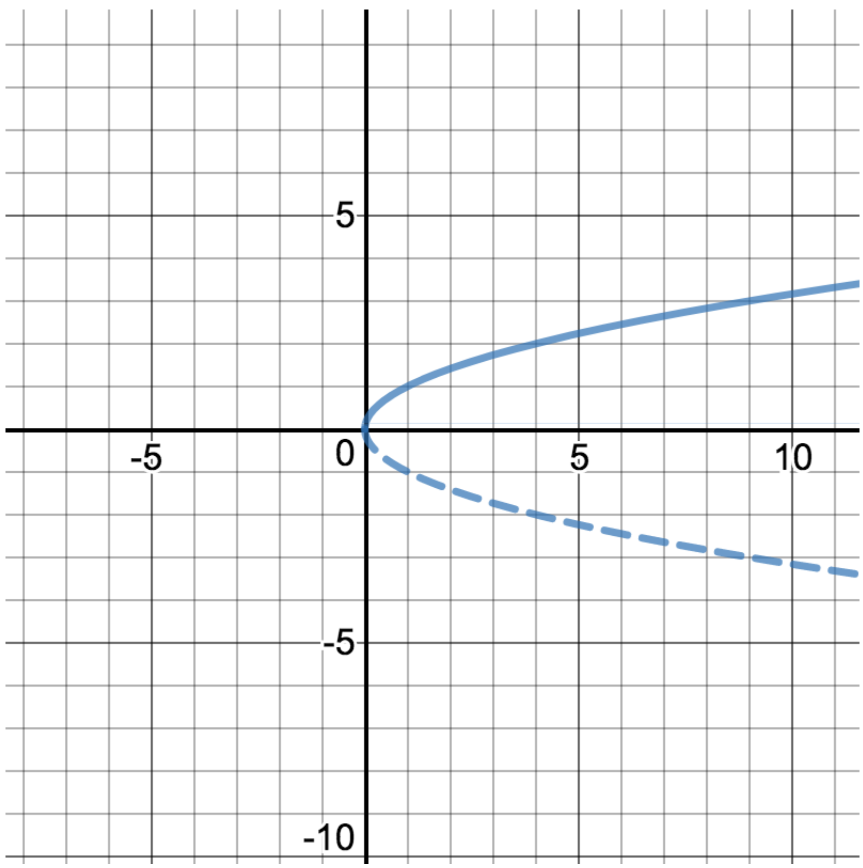
Another way is that these two solutions are in fact horizontal translations of each other. This can be understood by thinking that the cup of coffee starting at $C(0) = 160$ is somehow "ahead" of the other by some fixed amount of time. In order for the cup starting at $C(0) = 180$ to reach some fixed temperature, say 140, then it would take whatever time it takes to reach 160 + however long it took the cup of coffee starting at 160 to reach 140, so it will always be slower. This horizontal translation idea can also be seen by separating variables. Solutions will satisfy:

$$\int \frac{dC}{-0.4(C-70)} = \int dt = t + K$$

where the solutions are distinguished by the value of K, which can be seen here as a horizontal translation.

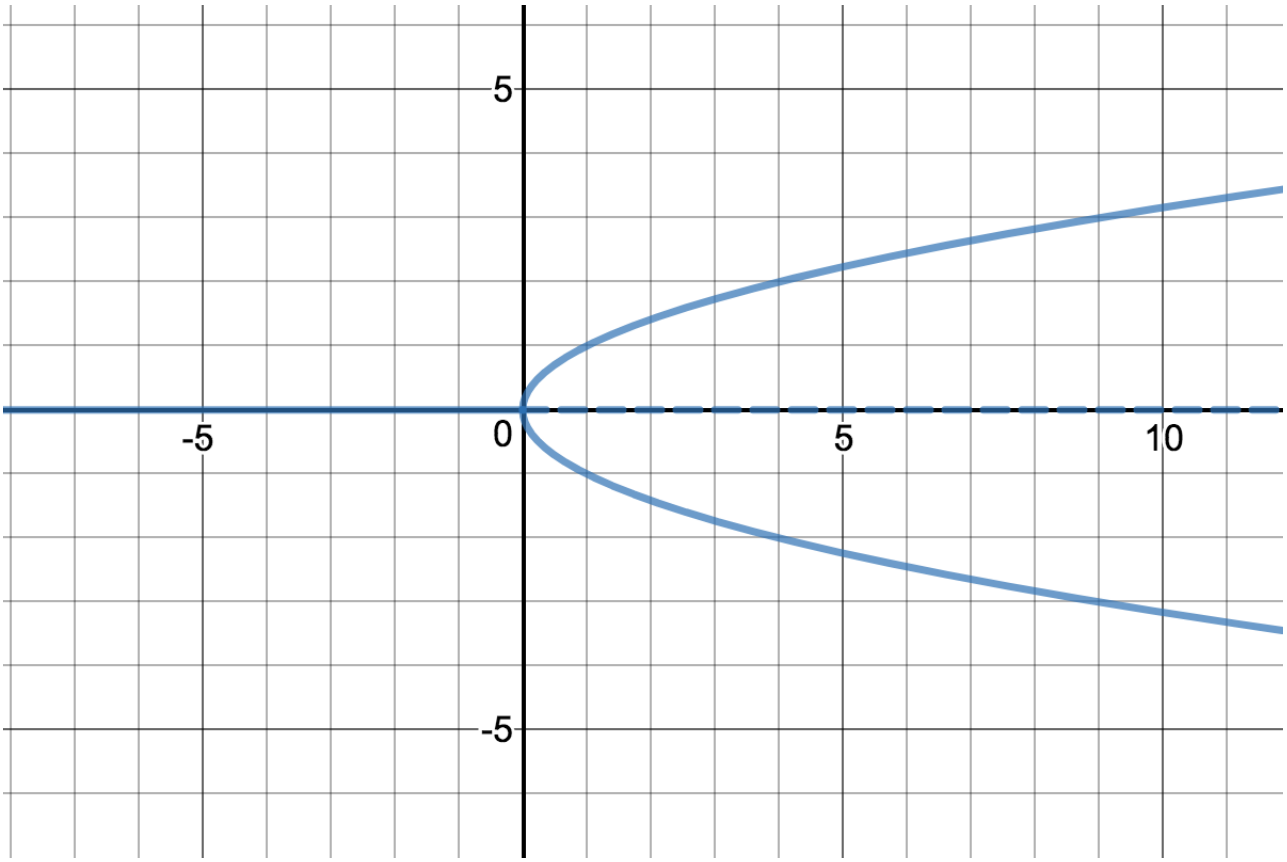
Answer to Question 2. (a) If a is any fixed positive number, then we see that $a - y^2$ has two solutions: $y = +\sqrt{a}$ and $y = -\sqrt{a}$. If we then look at the sign of y, we see that in between the two equilibria, $a - y^2$ is positive, so $y(t)$ is increasing. But outside the equilibria, $a - y^2$ is negative, so $y(t)$ is decreasing. Putting that together, we see that $y = +\sqrt{a}$ gives stable equilibria and $y = -\sqrt{a}$ gives unstable equilibria.
If a is any negative number, then $a - y^2 = 0$ has no solutions, so there will be no equilibria there. For the sake of completeness, the case of $a = 0$ has a single semi-stable equilibrium at $y = 0$.

All together, the graph will look something like:



Where the solid line represents stable equilibria, and the dashed line represents unstable equilibria.
This is what is known as a *saddle-node* bifurcation. As the parameter a changes from negative to positive, we see vastly different qualitative behavior in the system. This is getting a little outside the scope of this course, but is something you will probably see a lot of if you eventually take courses in nonlinear dynamics.

(b) The procedure for part **(b)** is very similar to the procedure for part **(a)** , except that now there is an extra equilibrium at $y = 0$ for *all* values of a.
For $a < 0$ there is only the equilibrium at $y = 0$, and it is stable in this region.
For $a > 0$, there are three equilibria, an unstable one at $y = 0$, and two stable ones at $y = \pm\sqrt{a}$ (We can figure out the stability from drawing the phase line like in **(a)** above).
The overall picture will look like this:



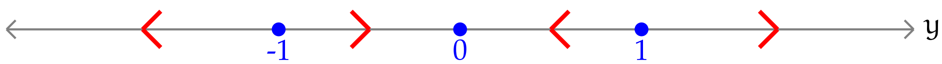
Where as before, the solid lines represent stable equilibria, and the dashed lines represent unstable equilibria. This system has what is known as a *pitchfork* bifurcation at $a = 0$ because of the shape of this graph, since it looks like a sideways pitchfork.

Answer to Question 3. (a) In order to have the given equilibria, we would want an $f(y)$ with exactly three zeroes, changing sign from positive to negative twice, and changing sign from negative to positive once.

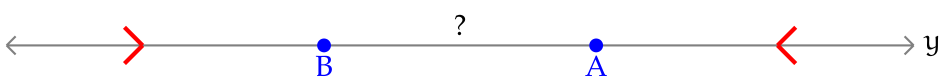
There are many possible answers that work here, but generally $f(y)$ that is cubic, has three distinct roots, and the leading term is positive will work. One such example is:

$$\frac{dy}{dt} = y^3 - y = y(y-1)(y+1)$$

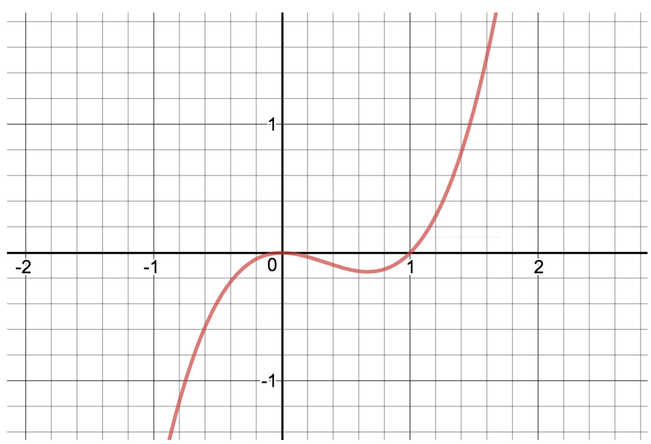
which would have a phase line:



(b) This one is actually impossible, as long as $f(y)$ is continuous. To see why, let's think about what the phase line would have to look like. In order for both equilibria to be stable, we would need solutions to be increasing for very negative y and increasing for very positive y. However, there's no way of setting the direction in between the two equilibria that will make both of them stable. In order to do so, $f(y)$ would have to go from negative to positive without going through zero, which is not possible with a continuous function f.



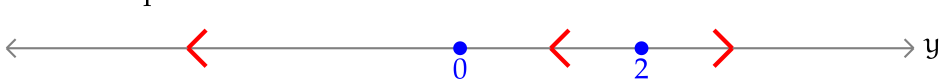
(c) In order to satisfy this, we would need a function $f(y)$ that crosses zero going from positive to negative exactly once, and that also touches zero without changing sign once (like the graph of $f(x) = x^2$).
I.e. the graph of $f(y)$ vs y would need to look like:



An example that would match this is:

$$\frac{dy}{dt} = (y-1)y^2$$

which has phase line:



Answer to Question 4. (a) Autonomous ODEs are of the form $\frac{dy}{dt} = f(y)$. So we'll want to come up with some sort of relation like this that is satisfied by $y(t) = e^{2t+1}$
So

$$\frac{dy}{dt} = 2e^{2t+1}$$

is a differential equation with the given solution, but it's not the final answer since it's not autonomous. To get an autonomous equation, we'll need to rewrite the right hand side in terms of y, which we can do as:

$$\frac{dy}{dt} = 2y$$

Which we hopefully recognize by now has solutions of the form $y(t) = Ce^{2t}$, which includes the given solution.

(b) This part is trickier than part **(a)** . There might be multiple ways of solving this, but the best one I know of is to realize that all the solutions to a given first order ODEs always look like:

$$y(t) = \text{ something involving t and a constant C}$$

So we want to come up with something of that form that matches both of the given solutions. The best one that I know of is to make the somewhat clever guess that solutions should look like:

$$y(t) = 1 + Ce{-}t$$

and then follow a similar process to part **(a)** .
So we calculate

$$\frac{dy}{dt} = -Ce{-}t$$

Which we can rewrite in terms of just y as

$$\frac{dy}{dt} = 1 - y$$