Math 2930 Worksheet
Final Exam Review

Week 14
November 30th, 2017

Question 1. (*) Solve the initial value problem

$$y' - y = 2xe^x, \qquad y(0) = 1$$

Question 2. (*) Consider the differential equation: $y' = y - y^3$.

(a) Find the equilibrium solutions and determine which of these solutions are asymptotically stable, semistable, and unstable.

(b) Draw the phase line and sketch several solution curves in the ty-plane for $t > 0$.

(c) Assuming that the solution $y(t)$ has the initial value $y(0) = -\frac{1}{2}$, compute the limit of $y(t)$ as $t \to +\infty$.

Question 3. (*) Find the general solution of the differential equation

$$\frac{dy}{dx} = \frac{x + 3y}{x - y}$$

Question 4. (*) Consider a uniform rod of length L with initial temperature distribution given by $u(x, 0) = \sin(\pi x/L)$, $0 \le x \le L$. Assuming that both ends of the rods are insulated,

(a) Find the temperature $u(x, t)$ for $t > 0$;

(b) Determine the steady state temperature as $t \to +\infty$

(c) Describe briefly how the temperature in the rod changes as time progresses.

Question 5. (*) **(a)** The 2-dimensional wave equation is given by

$$u_{tt} = a^2 \left(u_{xx} + u_{yy}\right) \tag{1}$$

Assuming that the solution of (1) has the form $u(x, y, t) = X(x)Y(y)T(t)$, find the ordinary differential equations satisfied by the functions $X(x)$, $Y(y)$, and $T(t)$.

(b) In polar coordinates, the 2D wave equation is written as

$$u_{tt} = a^2 \left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\right) \tag{2}$$

Assuming that the solution of (2) has the form $u(r, \theta, t) = R(r)\Theta(\theta)T(t)$, find the ordinary differential equations satisfied by the functions $R(r)$, $\Theta(\theta)$ and $T(t)$.

Question 6. (*) The Neumann problem for the Laplace equation in the interior of the circle $r = a$ is given by

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \qquad 0 \le r < a, \qquad 0 \le \theta < 2\pi$$
$$u_r(a, \theta) = f(\theta), \qquad 0 \le \theta < 2\pi$$

(a) Using the method of separation of variables, find the solution to this problem.

(b) What condition should one impose on the function $f(\theta)$ for this problem to be solvable?

Answer to Question 1. This is a first-order linear equation, so we can solve it using integrating factors. (You could also use something like method of undetermined coefficients, but this would be harder).
We have the equation

$$y' - y = 2xe^x$$

Multiplying both sides by $e^{\int -1dx} = e^{-x}$,

$$e^{-x}y' - e^{-x}y = 2x$$

and then using the product rule,

$$(e^{-x}y)' = 2x$$

Integrating both sides and then solving for y,

$$\begin{aligned}\int (e^{-x}y)'\,dx &= \int 2x dx \\ e^{-x}y &= x^2 + C \\ y &= x^2e^x + Ce^x\end{aligned}$$

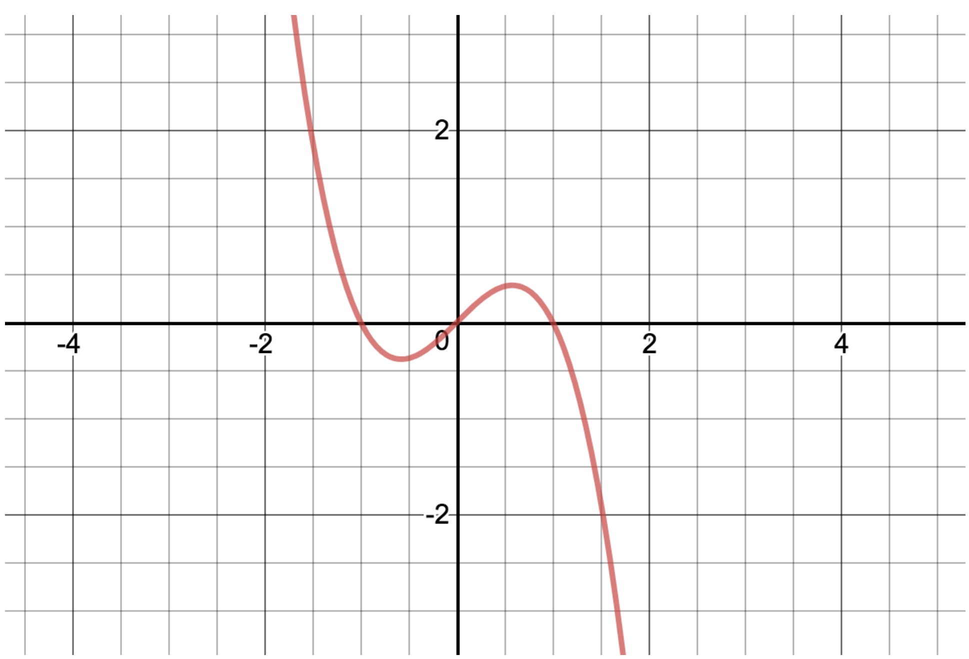
Now we plug in the initial condition to find C:

$$\begin{aligned}y(0) = 0^2(1) + C(1) &= 1 \\ C &= 1\end{aligned}$$

so the final answer is:

$$\boxed{y = x^2e^x + e^x}$$

Answer to Question 2.
(a) Since this is an autonomous equation, we'll first look at the graph of $\frac{dy}{dt}$ (vertical axis) vs y (horizontal axis):



The equilibria occur where this graph crosses the horizontal axis, these are the values of y where $\frac{dy}{dt} = 0$. Solving for them,

$$\frac{dy}{dt} = y - y^3 = 0$$
$$y(1 - y^2) = y(1 - y)(1 + y) = 0$$
$$y = 0, \quad 1, \quad -1$$

To figure out whether these equilibrium solutions are stable, unstable, or semistable, we will look at the sign of $\frac{dy}{dt}$ nearby.
For $y = -1$, we see that for values of y less than -1, $\frac{dy}{dt}$ if positive, so solutions are increasing. For values of y slightly greater than -1, we see $\frac{dy}{dt}$ is negative, so solutions are decreasing. Since solutions below $y = -1$ are increasing and solutions above are decreasing, we get that

$y = -1$ is a *stable* equilibrium

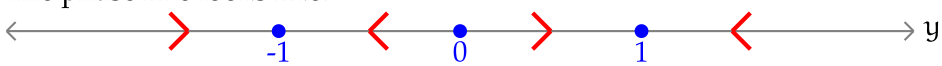
For $y = 0$, solutions slightly below are decreasing and solutions slightly above are increasing, so

$y = 0$ is an *unstable* equilibrium

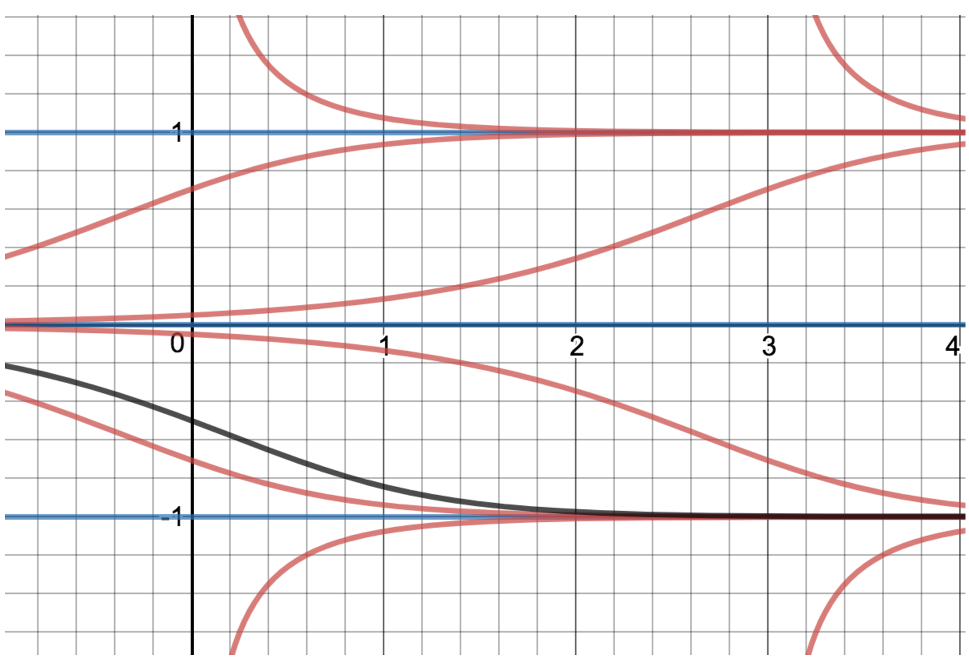
For $y = 1$, solutions slightly below are increasing and solutions slightly above are decreasing, so

$y = 1$ is a *stable* equilibrium

(b)
The phase line looks like:



and the solutions look like:



Here the equilibria are in blue, and solutions are in red and black.

If the initial condition is $y(0) = -\frac{1}{2}$, then the solution $y(t)$ will approach the nearest stable equilibrium at $y = -1$.
This specific solution $y(t)$ is the one graphed in black above.

Answer to Question 3. This equation can be solved by noticing that the right hand side can be written as a function of y/x only.
Factoring x out of both the numerator and denominator,

$$\frac{dy}{dx} = \frac{x+3y}{x-y} = \frac{x\left(1+3\frac{y}{x}\right)}{x\left(1-\frac{y}{x}\right)} = \frac{1+3\left(\frac{y}{x}\right)}{1-\left(\frac{y}{x}\right)}$$

So this means that we can solve the equation by using the substitution

$$v = \frac{y}{x} \quad \text{which can also be written as} \quad y = vx$$

Using the product rule,

$$\frac{d}{dx}[y] = \frac{d}{dx}[vx] = \frac{dv}{dx}x + v$$

So our differential equation becomes

$$\frac{dv}{dx}x + v = \frac{1+3v}{1-v}$$

This new equation is in fact separable:

$$\frac{dv}{dx}x = \frac{1+3v}{1-v} - v = \frac{1+2v+v^2}{1-v} = \frac{(1+v)^2}{1-v}$$

$$\int \frac{(1-v)}{(1+v)^2}dv = \int \frac{1}{x}dx$$

This integral on the left hand side can be computed using partial fractions, but (in my opinion at least) an easier way is to use another substitution of

$$u = v + 1$$

Now we have

$$\int \frac{(2-u)}{u^2}du = \int \frac{1}{x}dx$$

$$\int \left(\frac{2}{u^2} - \frac{1}{u}\right)du = \int \frac{1}{x}dx$$

$$\frac{-2}{u} - \ln|u| = \ln|x| + C$$

Now, we undo the substitutions,

$$\frac{-2}{v+1} - \ln|v+1| = \ln|x| + C$$

$$\frac{-2}{\frac{y}{x}+1} - \ln\left|\frac{y}{x}+1\right| = \ln|x| + C$$

which can be simplified slightly to

$$\frac{-2x}{x+y} = \ln|x| + \ln\left|\frac{y}{x}+1\right| + C$$

$$\boxed{\frac{2x}{x+y} + \ln|x+y| = C}$$

Answer to Question 4. The heat equation describing the temperature distribution of the rod is:

$$u_t = a u_{xx}$$

The ends of the rod being insulated means that our boundary conditions are:

$$u_x(0,t) = u_x(L,t) = 0$$

So using separation of variables, we look for solutions of the form:

$$u(x,t) = X(x)T(t)$$

Plugging this into the heat equation and separating variables,

$$XT' = aX''T$$
$$\frac{T'}{aT} = \frac{X''}{X}$$

Since the left hand side depends only on t and the right hand side depends only on x, they must both be equal to the same constant, which we will call $-\lambda$:

$$\frac{T'}{aT} = \frac{X''}{X} = -\lambda$$

which gives us two ODEs for $X(x)$ and $T(t)$:

$$X'' + \lambda X = 0$$
$$T' + a\lambda T = 0$$

We can also convert the boundary conditions on $u(x,t)$ into boundary conditions on $X(x)$ as follows:

$$u_x(0,t) = X'(0)T(t) = 0 \qquad \Longrightarrow \qquad X'(0) = 0$$
$$u_x(L,t) = X'(L)T(t) = 0 \qquad \Longrightarrow \qquad X'(L) = 0$$

So we are looking for nontrivial solutions to the boundary value problem

$$X'' + \lambda X = 0, \qquad X'(0) = X'(L) = 0$$

For $\lambda < 0$, we get only the trivial solution $X = 0$.
For $\lambda = 0$, we get the general solution is:

$$X(x) = c_1 x + c_2$$

Plugging in the boundary conditions, we get that $X(x) = c_2$ is a solution for any constant c_2.

For $\lambda > 0$, we get a general solution of:

$$X(x) = c_1 \cos\left(\sqrt{\lambda}x\right) + c_2 \sin\left(\sqrt{\lambda}x\right)$$

Plugging in the first boundary condition,

$$\begin{aligned} X'(0) = -\sqrt{\lambda}c_1 \sin(0) + \sqrt{\lambda}c_2 \cos(0) &= 0 \\ \sqrt{\lambda}c_2 &= 0 \\ c_2 &= 0 \end{aligned}$$

and the second boundary condition, looking for non-trivial solutions,

$$\begin{aligned} X'(L) == \sqrt{\lambda}c_1 \sin\left(\sqrt{\lambda}L\right) &= 0 \\ \sin\left(\sqrt{\lambda}L\right) &= 0 \\ \sqrt{\lambda}L &= n\pi, && n = 1, 2, 3... \\ \lambda &= \left(\frac{n\pi}{L}\right)^2, && n = 1, 2, 3... \end{aligned}$$

So, along with $\lambda = 0$, these are the eigenvalues, and they have corresponding eigenfunctions:

$$X_n(x) = c_n \cos\left(\frac{n\pi x}{L}\right), \qquad n = 1, 2, 3...$$

where c_n could be any constant.
Now to solve for $T(t)$ with these values of λ. For $\lambda = 0$, our equation for T is:

$$T'(t) = 0$$

the solutions to which are that T is constant.
For $\lambda = (n\pi/L)^2$, our equation for T is:

$$T' + \left(\frac{n\pi a}{L}\right)^2 T = 0$$

the solutions to which are:

$$T_n(t) = c_n e^{-\left(\frac{n\pi a}{L}\right)^2 t}$$

So putting this together, our solution to the heat equation is:

$$u(x,t) = \sum_{n=0}^{\infty} X_n(x)T_n(t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi a}{L}\right)^2 t}$$

Plugging in the initial conditions,

$$u(x,0) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi x}{L}\right) = \sin\left(\frac{\pi x}{L}\right), \qquad 0 < x < L$$

So this means we want the c_n to be given by the coefficients of the cosine series for $\sin(x)$. These are given by the formulas:

$$\begin{aligned} c_0 &= \frac{2}{L}\int_0^L \sin\left(\frac{\pi x}{L}\right) dx \\ c_n &= \frac{2}{L}\int_0^L \sin\left(\frac{\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx \end{aligned}$$

(*Note*: Many students are tempted to say here that these integrals are then zero because sin() and cos() are orthogonal. But really, they are just orthogonal on the interval $[-L, L]$, whereas here we are only integrating on $[0, L]$, over which the integrals are not orthogonal.)
We can then compute c_0 as follows:

$$c_0 = \frac{2}{L}\int_0^L \sin\left(\frac{\pi x}{L}\right) dx$$
$$c_0 = \frac{2}{L}\left[-\frac{L}{\pi}\cos\left(\frac{\pi x}{L}\right)\right]\Bigg|_0^L$$
$$c_0 = \frac{-2}{\pi}\left[\cos(\pi) - \cos(0)\right] = \frac{4}{\pi}$$

The integral for c_n is more difficult (and personally I would not expect students to be able to come up with it on their own during an exam). It can be done by using trig identities to change the product of trig functions into a sum.
The angle addition formulas tell us that:

$$\sin\left(\frac{(n+1)\pi x}{L}\right) = \sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right) + \cos\left(\frac{\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$$
$$\sin\left(\frac{(n-1)\pi x}{L}\right) = \sin\left(\frac{-\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right) + \cos\left(\frac{-\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$$
$$= -\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right) + \cos\left(\frac{\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$$

So we can combine the two equations above to get:

$$\frac{1}{2}\left[\sin\left(\frac{(n+1)\pi x}{L}\right) - \sin\left(\frac{(n-1)\pi x}{L}\right)\right] = \sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)$$

For the special case of c_1, this $n-1$ term will be zero, meaning we have to handle it on its own as follows:

$$c_1 = \frac{2}{L}\int_0^L \sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi x}{L}\right) dx$$
$$c_1 = \frac{1}{L}\int_0^L \sin\left(\frac{2\pi x}{L}\right) dx$$
$$c_1 = \frac{1}{L}\left[\frac{L}{2\pi}\cos\left(\frac{2\pi x}{L}\right)\right]\Bigg|_0^L = \frac{1}{2\pi}\left[\cos(2\pi) - \cos(0)\right] = 0$$

Now we can calculate the rest of the c_n as:

$$
\begin{aligned}
c_n &= \frac{2}{L}\int_0^L \sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx, \qquad n = 2,3,4...\\
c_n &= \frac{1}{L}\int_0^L \left[\sin\left(\frac{(n+1)\pi x}{L}\right) - \sin\left(\frac{(n-1)\pi x}{L}\right)\right]dx\\
c_n &= \frac{1}{L}\left[\frac{-L}{(n+1)\pi}\cos\left(\frac{(n+1)\pi x}{L}\right) + \frac{L}{(n-1)\pi}\cos\left(\frac{(n-1)\pi x}{L}\right)\right]\Bigg|_0^L\\
c_n &= \frac{-1}{(n+1)\pi}\left[\cos((n+1)\pi) - \cos(0)\right] + \frac{1}{(n-1)\pi}\left[\cos((n-1)\pi) - \cos(0)\right]\\
c_n &= \frac{-1}{(n+1)\pi}\left[(-1)^{n+1} - 1\right] + \frac{1}{(n-1)\pi}\left[(-1)^{n-1} - 1\right]\\
c_n &= \frac{2}{\pi(n^2-1)}\left[(-1)^{n+1} - 1\right], \qquad n = 2,3,4...
\end{aligned}
$$

Which we could rewrite all of the c_n (including c_0 and c_1) in the form:

$$
\boxed{c_n = \begin{cases} 0, & n = \text{ odd} \\ \frac{-4}{\pi(n^2-1)}, & n = \text{ even} \end{cases}}
$$

which are the coefficients of the solution:

$$
\boxed{u(x,t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi a}{L}\right)^2 t}}
$$

(b) If we look at our solution above, we see that we have a constant term in front, and all the other terms have some sort of negative exponential in t attached to them. So as $t \to \infty$, all of those terms in the infinite sum will approach zero, leaving us with just the constant term. Formally,

$$
\lim_{t\to\infty} u(x,t) = \lim_{t\to\infty}\left[\frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi a}{L}\right)^2 t}\right] = \frac{2}{\pi} + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi x}{L}\right)\lim_{t\to\infty}\left[e^{-\left(\frac{n\pi a}{L}\right)^2 t}\right]
$$

$$
\boxed{\lim_{t\to\infty} u(x,t) = \frac{2}{\pi}}
$$

is the steady state solution.

(c) At $t = 0$, the solution is given by the initial condition:

$$
u(x,0) = \sin\left(\frac{\pi x}{L}\right)
$$

Over time, the solution will then steadily converge towards the steady state solution:

$$
\lim_{t\to\infty} u(x,t) = \frac{2}{\pi}
$$

Because the ends are insulated, the total amount of heat (i.e. the average temperature) will remain the same at all times t. So $u(x,t)$ will start out with the initial condition, and then "smooth out" to the constant steady-state solution.

Answer to Question 5. (a) We look for a solution of the form

$$u(x,y,t) = X(x)Y(y)T(t)$$

plugging this into the 2D wave equation, we get:

$$XYT'' = a^2(X''YT + XY''T)$$

dividing both sides by XYT,

$$\frac{T''}{T} = a^2\left(\frac{X''}{X} + \frac{Y''}{Y}\right)$$

Since the left hand side depends only on t, and the right hand side does not depend on t, both sides must be equal to the same constant λ:

$$\frac{T''}{T} = a^2\left(\frac{X''}{X} + \frac{Y''}{Y}\right) = \lambda$$

We can then use this to get an ODE for $T(t)$:

$$\boxed{T'' - \lambda T = 0}$$

We also have the equation

$$a^2\left(\frac{X''}{X} + \frac{Y''}{Y}\right) = \lambda$$

We can then isolate the X terms, getting:

$$\frac{X''}{X} = \frac{\lambda}{a^2} - \frac{Y''}{Y}$$

since the left hand side depends only on x, and the right hand side depends only on y, they both must be equal to the same constant, which we will call μ:

$$\frac{X''}{X} = \frac{\lambda}{a^2} - \frac{Y''}{Y} = \mu$$

Which we can rearrange into the following ODEs for $X(x)$ and $Y(y)$:

$$\boxed{\begin{aligned} X'' - \mu X = 0 \\ Y'' - \left(\frac{\lambda}{a^2} - \mu\right)Y = 0 \end{aligned}}$$

(b) We look for a solution of the form:

$$u(r,\theta,t) = R(r)\Theta(\theta)T(t)$$

Plugging this into the PDE, we get:

$$R\Theta T'' = a^2\left(R''\Theta T + \frac{1}{r}R'\Theta T + \frac{1}{r^2}R\Theta''T\right)$$

Dividing everything by $R\Theta T$ leaves:

$$\frac{T''}{T} = a^2\left(\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta}\right)$$

Since the left hand side depends only on t, and the right hand side does not depend on t, both sides must be equal to the same constant, which we will call λ:

$$\frac{T''}{T} = a^2\left(\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta}\right) = \lambda$$

From which we can isolate an ODE for $T(t)$:

$$\boxed{T'' - \lambda T = 0}$$

Now, we are still left with:

$$a^2\left(\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta}\right) = \lambda$$

Multiplying everything by $\frac{r^2}{a^2}$,

$$r^2\frac{R''}{R} + r\frac{R'}{R} + \frac{\Theta''}{\Theta} = \frac{r^2\lambda}{a^2}$$

Separating the r and θ terms,

$$r^2\frac{R''}{R} + r\frac{R'}{R} - \frac{r^2\lambda}{a^2} = \frac{-\Theta''}{\Theta}$$

Since the left hand side depends only on r, and the right hand side only on θ, they must equal the same constant μ:

$$r^2\frac{R''}{R} + r\frac{R'}{R} - \frac{r^2\lambda}{a^2} = \frac{-\Theta''}{\Theta} = \mu$$

From which we can get ODEs for $R(r)$ and $\Theta(\theta)$:

$$\boxed{\Theta'' + \mu\Theta = 0}$$

$$\boxed{r^2R'' + rR' - \left(\frac{r^2\lambda}{a^2} + \mu\right)R = 0}$$

Answer to Question 6. (a) For the method of separation of variables, we look for solutions of the form:

$$u(r\theta) = R(r)\Theta(\theta)$$

Plugging this into Laplace's equation in polar coordinates,

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = 0$$

Separating variables,

$$\frac{r^2R''}{R} + \frac{rR'}{R} = -\frac{\Theta''}{\Theta}$$

Since the left hand side depends only on r, and the right hand side only on θ, they both must be equal to the same constant λ:

$$\frac{r^2R''}{R} + \frac{rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda$$

We can use this to get the following ODEs for $R(r)$ and $\Theta(\theta)$:

$$\Theta'' + \lambda\Theta = 0$$
$$r^2R'' + rR' - \lambda R = 0$$

In order for our solution to be well-defined in polar coordinates, we want to make sure that our solution is the same when we increase θ by 2π, as this is just traveling around a circle back to the same point. This means that $u(r,\theta)$ should be periodic in θ with period 2π.
More precisely, we need that for all angles θ:

$$u(r,\theta) = u(r,\theta + 2\pi)$$
$$R(r)\Theta(\theta) = R(r)\Theta(\theta + 2\pi)$$
$$\Theta(\theta) = \Theta(\theta + 2\pi)$$

This will serve in the role of boundary conditions for our ODE for $\Theta(\theta)$. In other words, we are looking for non-trivial solutions to:

$$\Theta'' + \lambda\Theta = 0, \qquad \Theta(\theta) = \Theta(\theta + 2\pi)$$

For $\lambda < 0$, our solutions for Θ are of the form:

$$\Theta(\theta) = c_1 e^{\sqrt{\lambda}\theta} + c_2 e^{-\sqrt{\lambda}\theta}$$

In order for this to be periodic,

$$\Theta(\theta) = \Theta(\theta + 2\pi)$$
$$c_1 e^{\sqrt{\lambda}\theta} + c_2 e^{-\sqrt{\lambda}\theta} = c_1 e^{\sqrt{\lambda}(\theta+2\pi)} + c_2 e^{-\sqrt{\lambda}(\theta+2\pi)}$$
$$c_1 e^{\sqrt{\lambda}\theta} + c_2 e^{-\sqrt{\lambda}\theta} = c_1 e^{\sqrt{\lambda}\theta} e^{2\pi\sqrt{\lambda}} + c_2 e^{-\sqrt{\lambda}\theta} e^{-2\pi\sqrt{\lambda}}$$

and matching like terms, we would need that:

$$c_1 = c_1 e^{2\pi\sqrt{\lambda}} \quad \text{and} \quad c_2 = c_2 e^{-2\pi\sqrt{\lambda}}$$

This only happens when $c_1 = c_2 = 0$, which is the trivial solution.
For $\lambda = 0$, our solutions for Θ are of the form:

$$\Theta(\theta) = c_1 + c_2\theta$$

In order for this to be periodic,

$$\Theta(\theta) = \Theta(\theta + 2\pi)$$
$$c_1 + c_2\theta = c_1 + c_2(\theta + 2\pi)$$
$$c_1 + c_2\theta = c_1 + c_2(2\pi) + c_2\theta$$
$$0 = c_2(2\pi)$$
$$c_2 = 0$$

So $c_2 = 0$, but c_1 could be any constant. This means that $\Theta_0(\theta) = c_1$ works for any constant c_1.
When $\lambda = 0$, the equation for R is:

$$r^2R'' + rR' = 0$$

This is an Euler equation, so looking for solutions of the form $R = r^m$,

$$\begin{aligned} m(m-1) + m &= 0 \\ m^2 &= 0 \end{aligned}$$

So we have a repeated root at $m = 0$. This corresponds to a solution of:

$$R_0(r) = c_1 + c_2 \ln(r)$$

Since the natural logarithm is not defined at the origin $r = 0$, if we want our solution u to be defined at $r = 0$, we need to set $c_2 = 0$, leaving:

$$R_0(r) = c_1$$

So for $\lambda = 0$, we have that R can also be any constant. Thus we have an eigenvalue-eigenfunction pair of:

$$\lambda = 0, \qquad u_0(r, \theta) = \frac{c_0}{2}$$

where c_0 could be any constant (I'm writing it this way since it will end up being useful later).
For $\lambda > 0$, our solutions for Θ are of the form:

$$\Theta(\theta) = A\cos\left(\sqrt{\lambda}\theta\right) + B\sin\left(\sqrt{\lambda}\theta\right)$$

In order for this to be periodic,

$$\begin{aligned} \Theta(\theta) &= \Theta(\theta + 2\pi) \\ A\cos\left(\sqrt{\lambda}\theta\right) + B\sin\left(\sqrt{\lambda}\theta\right) &= A\cos\left(\sqrt{\lambda}(\theta + 2\pi)\right) + B\sin\left(\sqrt{\lambda}(\theta + 2\pi)\right) \end{aligned}$$

Using the angle addition trig identities, the right hand side becomes:

$$= A\left[\cos\left(\sqrt{\lambda}\theta\right)\cos\left(2\pi\sqrt{\lambda}\right) - \sin\left(\sqrt{\lambda}\theta\right)\sin\left(2\pi\sqrt{\lambda}\right)\right] + B\left[\sin\left(\sqrt{\lambda}\theta\right)\cos\left(2\pi\sqrt{\lambda}\right) + \sin\left(2\pi\sqrt{\lambda}\right)\cos\left(\sqrt{\lambda}\theta\right)\right]$$

Grouping together the terms by θ,

$$= \left[A\cos\left(2\pi\sqrt{\lambda}\right) + B\sin\left(2\pi\sqrt{\lambda}\right)\right]\cos\left(\sqrt{\lambda}\theta\right) + \left[B\cos\left(2\pi\sqrt{\lambda}\right) - A\sin\left(2\pi\sqrt{\lambda}\right)\right]\sin\left(\sqrt{\lambda}\theta\right)$$

Comparing like terms, A and B should satisfy the equations:

$$\begin{aligned} A &= A\cos\left(2\pi\sqrt{\lambda}\right) + B\sin\left(2\pi\sqrt{\lambda}\right) \\ B &= B\cos\left(2\pi\sqrt{\lambda}\right) - A\sin\left(2\pi\sqrt{\lambda}\right) \end{aligned}$$

This is always true for $A = B = 0$, but this corresponds to the trivial solution. However, for values of λ where the cosine term above is 1 and the sine term is 0, this would be true for any A and B. More precisely, we want the values of λ where:

$$\begin{aligned} \cos\left(2\pi\sqrt{\lambda}\right) &= 1 \qquad \Longrightarrow \qquad 2\pi\sqrt{\lambda} = 2\pi n, \qquad n = 1, 2, 3... \\ \sin\left(2\pi\sqrt{\lambda}\right) &= 0 \qquad \Longrightarrow \qquad 2\pi\sqrt{\lambda} = \pi n, \qquad n = 1, 2, 3... \end{aligned}$$

Since we want *both* to be true, we take the more restrictive condition that:

$$2\pi\sqrt{\lambda} = 2\pi n, \qquad n = 1, 2, 3...$$
$$\sqrt{\lambda} = n, \qquad n = 1, 2, 3...$$
$$\lambda = n^2, \qquad n = 1, 2, 3...$$

So these are our eigenvalues, and they have corresponding eigenfunctions:

$$\Theta_n(\theta) = A\cos(n\theta) + B\sin(n\theta), \qquad n = 1, 2, 3...$$

Now we want to solve for $R(r)$ at these eigenvalues $(\lambda = n^2)$. We get an Euler equation:

$$r^2R'' + rR' - n^2R = 0$$

Looking for solutions of the form $R = r^m$, we plug this in and solve for m:

$$m(m-1) + m - n^2 = 0$$
$$m^2 - n^2 = 0$$
$$m^2 = n^2$$
$$m = \pm n$$

So $R(r)$ looks like:

$$R_n(r) = Ar^n + Br^{-n}$$

However, because we want R to be defined at $r = 0$, we then take $B = 0$, leaving:

$$R_n(r) = Ar^n$$

Putting this all together (and renaming some constants), our general solution is of the form:

$$u(r,\theta) = \sum_{n=0}^{\infty} R_n(r)\Theta_n(\theta)$$
$$u(r,\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} r^n\left[A_n\cos(n\theta) + B_n\sin(n\theta)\right]$$

Now we still have to apply the Neumann boundary conditions. Taking the partial derivative with respect to r,

$$u_r(r,\theta) = \sum_{n=1}^{\infty} nr^{n-1}\left[A_n\cos(n\theta) + B_n\sin(n\theta)\right]$$

Plugging in $r = a$, and setting it equal to $f(\theta)$

$$u_r(a,\theta) = \sum_{n=1}^{\infty} na^{n-1}\left[A_n\cos(n\theta) + B_n\sin(n\theta)\right] = f(\theta), \qquad 0 \le \theta < 2\pi$$

This means that we want $u_r(a,\theta)$ to match the Fourier series for $f(\theta)$ on the interval $[0, 2\pi]$. Using the formula for Fourier series coefficients, we get the following equations for A_n and B_n:

$$na^{n-1}A_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos(n\theta)d\theta$$
$$na^{n-1}B_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin(n\theta)d\theta$$

So our solution to the problem is:

$$\boxed{u(r,\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} r^n \left[A_n \cos(n\theta) + B_n \sin(n\theta)\right]}$$

where the coefficients are given by:

$$\boxed{\begin{aligned} A_n &= \frac{1}{na^{n-1}\pi}\int_0^{2\pi} f(\theta)\cos(n\theta)d\theta \\ B_n &= \frac{1}{na^{n-1}\pi}\int_0^{2\pi} f(\theta)\sin(n\theta)d\theta \\ c_0 &= \text{ any constant} \end{aligned}}$$

(b) In our solution to part **(a)** , when we were enforcing the boundary conditions on u_r, we got to an equation of the form:

$$u_r(a,\theta) = \sum_{n=1}^{\infty} na^{n-1}\left[A_n \cos(n\theta) + B_n \sin(n\theta)\right] = f(\theta), \qquad 0 \le \theta < 2\pi$$

and then went on to take a Fourier series expansion of $f(\theta)$. However, Fourier series usually have a constant term $\frac{c_0}{2}$. While our formula for $u(r,\theta)$ had this, when we took the derivative, this then makes sure that $u_r(r,\theta)$ does *not* have a constant term.
So for this problem to be solvable, we would need the $\frac{c_0}{2}$ constant term of the Fourier series for $f(\theta)$ to be zero. Rephrasing this in terms of a condition on f,

$$\boxed{\frac{1}{\pi}\int_0^{2\pi} f(\theta)d\theta = 0}$$