Math 2930 Worksheet
Wave Equation

Week 13
November 16th, 2017

Question 1. Consider the wave equation

$$a^2 u_{xx} = u_{tt}$$

in an infinite one-dimensional medium subject to the initial conditions

$$u(x,0) = 0$$
$$u_t(x,0) = g(x)$$

(a) Using the fact that the solution $u(x,t)$ can be written in the form $u(x,t) = F(x+at) + G(x-at)$, show that:

$$F(x) + G(x) = 0$$
$$aF'(x) - aG'(x) = g(x)$$

(b) Use the equations from part **(a)** to show that

$$2aF'(x) = g(x)$$

and therefore that

$$F(x) = \frac{1}{2a}\int_{x_0}^{x} g(\xi)d\xi + F(x_0)$$

where x_0 is arbitrary.

(c) Show that

$$G(x) = -\frac{1}{2a}\int_{x_0}^{x} g(\xi)d\xi - F(x_0)$$

(d) Show that

$$u(x,t) = \frac{1}{2a}\int_{x-at}^{x+at} g(\xi)d\xi$$

Question 2. For the wave equation in an infinite one-dimensional medium

$$a^2 u_{xx} = u_{tt}$$

We showed that the solution to the initial displacement problem is :

$$\begin{array}{c} u(x,0) = f(x) \\ u_t(x,0) = 0 \end{array} \quad \Longrightarrow \quad u(x,t) = \frac{1}{2}(f(x-at) + f(x+at))$$

The previous question showed that the solution to the initial velocity problem is :

$$\begin{array}{c} u(x,0) = 0 \\ u_t(x,0) = g(x) \end{array} \quad \Longrightarrow \quad u(x,t) = \frac{1}{2a}\int_{x-at}^{x+at} g(\xi)d\xi$$

Combine these two answers to show that the solution of the problem

$$\begin{aligned} a^2 u_{xx} &= u_{tt} \\ u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \end{aligned}$$

is

$$u(x,t) = \frac{1}{2}(f(x-at) + f(x+at)) + \frac{1}{2a}\int_{x-at}^{x+at} g(\xi)d\xi$$

Question 3. (*) A string is stretched and secured on the x-axis between the points $x = 0$ and $x = \pi$. If the transverse vibrations take place in a medium that imparts a resistance proportional to the instantaneous velocity, then the wave equation describing the vibrations takes on the form

$$u_{xx} = u_{tt} + 2\lambda u_t, \quad 0 < \lambda < 1, \quad t > 0 \tag{1}$$

(a) Changing the variable u to $v = e^{\lambda t} u$ in Equation (1), find the differential equation for the function $v = v(x, t)$.

(b) Determine the displacement $u(x, t)$ of the string at any time $t > 0$, provided the string starts vibrating from rest with the initial displacement $f(x)$.

[**Hint**: Use the result of Part **(a)** to simplify the initial boundary value problem in Part **(b)** .]

Question 4. Our derivation of d'Alembert's formula:

$$u(x,t) = \frac{1}{2}(f(x-at) + f(x+at))$$

assumed an infinitely-long one-dimensional medium (*i.e.* $-\infty < x < \infty$).

However, many practical problems do not take place on an infinite domain, but instead on an interval $[0, L]$ with boundary conditions at $x = 0$ and $x = L$.

Now, suppose that we have a problem for the wave equation:

$$a^2 u_{xx} = u_{tt}$$

and initial conditions

$$u(x,0) = f(x), \qquad u_t(x,0) = 0, \qquad 0 \le x \le L$$

and boundary conditions

$$u(0,t) = 0, \qquad u(L,t) = 0$$

Let $h(\xi)$ represent the initial displacement f in $[0, L]$, extended into $(-L, 0)$ as an odd functinon, and extended elsewhere as a periodic function of period 2L.
Show that:

$$u(x,t) = \frac{1}{2}(h(x-at) + h(x+at))$$

satsifies the wave equation, initial conditions *and* the boundary conditions.

This means that with this extension, the solution for the infinite string is also applicable to the finite case.

Answer to Question 1. (a) Our solution can be written as:

$$u(x,t) = F(x+at) + G(x-at)$$

So plugging in $t = 0$, we get

$$u(x,0) = \boxed{F(x) + G(x) = 0}$$

if we calculate u_t using the Chain Rule,

$$u_t(x,t) = aF'(x+at) - aG'(x-at)$$

Then plugging in $t = 0$, we get

$$u_t(x,0) = \boxed{aF'(x) - aG'(x) = g(x)}$$

(b) The first (boxed) equation above for can be re-arranged to show

$$F(x) = -G(x)$$

Plugging this into the second equation, we get

$$\boxed{2aF'(x) = g(x)}$$

Then dividing both sides by $2a$,

$$F'(x) = \frac{1}{2a} g(x)$$

and integrating both sides from x_0 to x, we get:

$$\int_{x_0}^{x} F'(\xi)d\xi = \frac{1}{2a}\int_{x_0}^{x} g(\xi)d\xi$$

$$F(x) - F(x_0) = \frac{1}{2a}\int_{x_0}^{x} g(\xi)d\xi$$

$$\boxed{F(x) = \frac{1}{2a}\int_{x_0}^{x} g(\xi)d\xi + F(x_0)}$$

(c) Since $F(x) = \frac{1}{2a}\int_{x_0}^{x} g(\xi)d\xi + F(x_0)$ and $F(x) = -G(x)$, we then get:

$$\boxed{G(x) = -\frac{1}{2a}\int_{x_0}^{x} g(\xi)d\xi - F(x_0)}$$

(d) We recall that our solution $u(x,t)$ is:

$$u(x,t) = F(x+at) + G(x-at)$$

So plugging $x + at$ and $x - at$ into our formulas for F and G above,

$$u(x,t) = F(x+at) + G(x-at)$$

$$u(x,t) = \frac{1}{2a}\int_{x_0}^{x+at} g(\xi)d\xi + F(x_0) - \frac{1}{2a}\int_{x_0}^{x-at} g(\xi)d\xi - F(x_0)$$

$$u(x,t) = \frac{1}{2a}\int_{x_0}^{x+at} g(\xi)d\xi + \frac{1}{2a}\int_{x-at}^{x_0} g(\xi)d\xi$$

$$\boxed{u(x,t) = \frac{1}{2a}\int_{x-at}^{x+at} g(\xi)d\xi}$$

Answer to Question 2. Let v be the solution to the initial displacement problem, *i.e.*

$$v(x,t) = \frac{1}{2}(f(x-at) + f(x+at))$$

then we know that v satisfies:

$$a^2 v_{xx} = v_{tt}, \qquad v(x,0) = f(x), \qquad v_t(x,0) = 0$$

Now let w be the solution to the initial velocity problem, *i.e.*

$$w(x,t) = \frac{1}{2a}\int_{x-at}^{x+at} g(\xi)d\xi$$

then we know that w satisfies:

$$a^2 w_{xx} = w_{tt}, \qquad w(x,0) = 0, \qquad w_t(x,0) = g(x)$$

Now we combine these two answers into:

$$\boxed{u(x,t) = v(x,t) + w(x,t) = \frac{1}{2}(f(x-at) + f(x+at)) + \frac{1}{2a}\int_{x-at}^{x+at} g(\xi)d\xi}$$

Then this solves the wave equation because:

$$a^2 u_{xx} = a^2(v+w)_{xx} = a^2 v_{xx} + a^2 w_{xx} = v_{tt} + w_{tt} = (v+w)_{tt} = u_{tt}$$

It solves the first initial condition:

$$u(x,0) = v(x,0) + w(x,0) = f(x) + 0 = f(x)$$

and the second initial condition:

$$u_t(x,0) = v_t(x,0) + w_t(x,0) = 0 + g(x) = g(x)$$

Answer to Question 3. (a) Instead of working with $v = e^{\lambda t}u$, it will actually be easier to work with:

$$u = e^{-\lambda t}v$$

Taking partial derivatives in x,

$$u_x = e^{-\lambda t}v_x$$
$$u_{xx} = e^{-\lambda t}v_{xx}$$

Taking partial derivatives in t using the Product Rule,

$$u_t = e^{-\lambda t}v_t - \lambda e^{-\lambda t}v$$
$$u_{tt} = e^{-\lambda t}v_{tt} - 2\lambda e^{-\lambda t}v_t + \lambda^2 e^{-\lambda t}v$$

The PDE in terms of u is:

$$u_{xx} = u_{tt} + 2\lambda u_t$$

So plugging things in to get this in terms of v:

$$e^{-\lambda t}v_{xx} = e^{-\lambda t}v_{tt} - 2\lambda e^{-\lambda t}v_t + \lambda^2 e^{-\lambda t}v + 2\lambda\left(e^{-\lambda t}v_t - \lambda e^{-\lambda t}v\right)$$
$$e^{-\lambda t}v_{xx} = e^{-\lambda t}v_{tt} - \lambda^2 e^{-\lambda t}v$$

$$\boxed{v_{xx} = v_{tt} - \lambda^2 v}$$

(b) In this problem, we will solve for $v(x,t)$ using our equation from part **(a)** instead of $u(x,t)$. So our PDE is:

$$v_{xx} = v_{tt} - \lambda^2 v$$

Our boundary conditions on u are:

$$u(0,t) = u(\pi,t) = 0$$

so writing these in terms of v:

$$e^{-\lambda t}v(0,t) = e^{-\lambda t}v(\pi,t) = 0$$
$$v(0,t) = v(\pi,t) = 0$$

And our initial conditions on u are:

$$u(x,0) = f(x), \qquad u_t(x,0) = 0$$

so writing these in terms of v:

$$u(x,0) = e^0 v(x,0) = f(x)$$
$$v(x,0) = f(x)$$

and

$$u_t(x,0) = e^0 v_t(x,0) - \lambda v(x,0)e^0 = 0$$
$$v_t(x,0) = \lambda v(x,0) = \lambda f(x)$$

(*Note*: be careful that even though u_t is zero at $t = 0$, this is not true of v_t)
Now for separation of variables. We look for solutions of the form

$$v(x,t) = X(x)T(t)$$

Plugging this into our PDE and separating variables,

$$X''T = T''X - \lambda^2 XT$$
$$\frac{X''}{X} = \frac{T''}{T} - \lambda^2$$

since the left hand side is a function of x only, and the right hand side is a function of t only, they must both be equal to the same constant μ:

$$\frac{X''}{X} = \frac{T''}{T} - \lambda^2 = \mu$$

Rearranging this into equations for X and T, we get:

$$X'' - \mu X = 0, \qquad T'' - (\mu + \lambda^2)T = 0$$

Then the boundary values on $v(x,t)$ become boundary values on $X(x)$:

$$X(0) = X(\pi) = 0$$

There are now three different cases we have to consider:

$$\mu > 0, \qquad \mu = 0, \qquad \mu < 0$$

If $\underline{\mu > 0}$, $X(x)$ takes the form:

$$X(x) = c_1 e^{\sqrt{\mu}x} + c_2 e^{-\sqrt{\mu}x}$$

Plugging in the first boundary condition,

$$X(0) = c_1 + c_2 = 0$$

and plugging in the second boundary condition,

$$X(\pi) = c_1 e^{\sqrt{\mu}\pi} + c_2 e^{-\sqrt{\mu}\pi} = 0$$

and the only way of satisfying both of these equations is if $c_1 = c_2 = 0$, which leads to the trivial solution of $u = 0$.

If $\underline{\mu = 0}$, $X(x)$ takes the form:

$$X(x) = c_1 + c_2 x$$

Plugging in the first boundary condition,

$$X(0) = c_1 = 0$$

and plugging in the second boundary condition,

$$X(\pi) = c_1 + c_2\pi = 0$$

and the only way of satisfying both of these equations is if $c_1 = c_2 = 0$, which again leads to the trivial solution of $u = 0$.

Now, if $\underline{\mu < 0}$, then we can write $\mu = -\alpha^2$ for some α, and $X(x)$ takes the form:

$$X(x) = c_1 \cos(\alpha x) + c_2 \sin(\alpha x)$$

Plugging in the first boundary condition,

$$X(0) = c_1 = 0$$

and plugging in the second boundary condition,

$$X(\pi) = c_2 \sin(\alpha\pi) = 0$$

if $c_2 = 0$, we get the trivial solution $u = 0$, but we can get nontrivial solutions if:

$$\begin{aligned} \sin(\alpha\pi) &= 0 \\ \alpha\pi &= n\pi, \quad n = 1, 2, 3... \end{aligned}$$

Thus we get nontrivial solutions for X when:

$$\lambda = -n^2$$
$$X_n(x) = c_n \sin(nx)$$

For these values of λ, we then get in our equation for $T(t)$ that:

$$T'' + (n^2 - \lambda^2)T = 0$$

Since $\lambda < 1$ was stated in the problem, this means that $n^2 - \lambda^2$ is positive for all n, and so $T(t)$ takes on the form:

$$T_n(t) = c_1 \cos\left(\sqrt{n^2 - \lambda^2}t\right) + c_2 \sin\left(\sqrt{n^2 - \lambda^2}t\right)$$

So our fundamental solutions are:

$$v_n(x,t) = X_n(x)T_n(t) = \sin(nx)\left(A_n \cos\left(\sqrt{n^2 - \lambda^2}t\right) + B_n \sin\left(\sqrt{n^2 - \lambda^2}t\right)\right)$$

And our general solution is:

$$v(x,t) = \sum_{n=1}^{\infty} c_n v_n(x,t) = \sum_{n=1}^{\infty} c_n \sin(nx)\left(A_n \cos\left(\sqrt{n^2 - \lambda^2}t\right) + B_n \sin\left(\sqrt{n^2 - \lambda^2}t\right)\right)$$

Now we will have to find the coefficients A_n and B_n to match the initial conditions.
If we plug in $t = 0$, then we get:

$$v(x,0) = \sum_{n=1}^{\infty} A_n \sin(nx) = f(x)$$

which means that A_n are the coefficients of the sine series for $f(x)$, so the A_n's are given by:

$$\boxed{A_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx}$$

For the other initial condition, we get:

$$v_t(x,0) = \sum_{n=1}^{\infty} B_n \sqrt{n^2 - \lambda^2}\sin(nx) = \lambda f(x)$$

which means that $\sqrt{n^2 - \lambda^2}B_n$ are the coefficients of the sine series for $\lambda f(x)$, so the B_n's are given by:

$$\sqrt{n^2 - \lambda^2}B_n = \frac{2}{\pi}\int_0^{\pi} \lambda f(x)\sin(nx)\,dx$$

$$\boxed{B_n = \frac{2\lambda}{\pi\sqrt{n^2 - \lambda^2}}\int_0^{\pi} f(x)\sin(nx)\,dx}$$

So with these coefficients, the solution of the PDE is:

$$\boxed{u(x,t) = e^{-\lambda t}v(x,t) = \sum_{n=1}^{\infty} c_n \sin(nx)e^{-\lambda t}\left(A_n \cos\left(\sqrt{n^2 - \lambda^2}t\right) + B_n \sin\left(\sqrt{n^2 - \lambda^2}t\right)\right)}$$

Answer to Question 4. Since

$$u(x,t) = \frac{1}{2}(h(x-at) + h(x+at))$$

is in the form of $F(x-at)+G(x+at)$, we have already shown that this satisfies the wave equation. As for the initial conditions, if we plug in $t = 0$, we get:

$$u(x,0) = \frac{1}{2}(h(x) + h(x)) = h(x)$$

which is equivalent to $f(x)$ on the interval $[0, L]$ by our process of creating $h(x)$. So this initial condition is satisfied.

For the second initial condition,

$$u_t(x,0) = \frac{1}{2}(-ah'(x) + ah'(x)) = 0$$

so this satisfies our second initial condition.

For the first boundary condition, we plug in $x = 0$

$$u(0,t) = \frac{1}{2}(h(-at) + h(at))$$

Since $h(x)$ is odd by construction, this means that $h(-at) = -h(at)$, and then:

$$u(0,t) = \frac{1}{2}(-h(at) + h(at)) = 0$$

so the first boundary condition is always satisfied.

For the second boundary condition, we plug in $x = L$

$$u(L,t) = \frac{1}{2}(h(L-at) + h(L+at))$$

Since $h(x)$ is periodic with period $2L$ by construction, it follows that $h(L-at) = h(L-at-2L)$ and then:

$$u(L,t) = \frac{1}{2}(h(L-at-2L) + h(L+at)) = \frac{1}{2}(h(-L-at) + h(L+at))$$

and since $h(x)$ is odd by construction, this means that $h(-L-at) = -h(L+at)$, and then:

$$u(L,t) = \frac{1}{2}(-h(L+at) + h(L+at)) = 0$$

so the second boundary condition is always satisfied.