Math 2930 Worksheet
Heat Equation

Week 12
November 9th, 2017

Question 1. Consider the heat equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$$

where a is constant.

(a) Assuming that $u(x,t) = X(x)T(t)$, find ordinary differential equations that are satisfied by $X(x)$ and $T(t)$ using separation of variables.

(b) Given the boundary conditions for u:

$$u_x(0,t) = u_x(L,t) = 0 \qquad \text{for all } t \geq 0$$

Should these place any boundary or initial conditions on $X(x)$ and/or $T(t)$?
If so, what are they?

(c) Solve the eigenvalue problem for $X(x)$ corresponding to what you found in **(b)** and find the fundamental solutions $u_n(x,t)$ of this heat equation.

(d) Take a linear combination of all of the fundamental solutions $u_n(x, t)$ to get the general solution $u(x, t)$ of this heat equation.

(e) Given the initial conditions:

$$u(x, 0) = f(x)$$

Find the solution $u(x, t)$ to the heat equation with these initial conditions

Question 2. (a) The heat conduction equation in two space dimensions is:

$$\alpha^2(u_{xx} + u_{yy}) = u_t$$

Assuming that $u(x, y, t) = X(x)Y(y)T(t)$, show that $X(x)$, $Y(y)$, and $T(t)$ satisfy the following ordinary differential equations:

$$X'' + \mu^2 X = 0$$
$$Y'' + (\lambda^2 - \mu^2)Y = 0$$
$$T' + \alpha^2\lambda^2 T = 0$$

where λ and μ are constants.

(b) The heat conduction equation in two space dimensions may also be expressed in polar coordinates as:

$$\alpha^2\left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\right) = u_t$$

Assuming that $u(r, \theta, t) = R(r)\Theta(\theta)T(t)$, find ordinary differential equations that are satisfied by $R(r)$, $\Theta(\theta)$, and $T(t)$.

Question 3. If we think of a solution $u(x, t)$ as describing the density of "heat particles" at location x and time t in a metal rod, then the quantity:

$$E(t) = \int_0^L u(x, t)dx$$

can be thought of as adding up the number of "heat particles" in the metal rod at a given time t.

(a) Show that if the ends of the metal rod are insulated, *i.e.*

$$u_x(0, t) = 0, \qquad u_x(L, t) = 0$$

then $E(t)$ is constant. [1]
(Hint: what is $\frac{dE}{dt}$?)

(b) Show that if the ends of the metal rod are connected together, *i.e.*

$$u(0, t) = u(L, t), \qquad u_x(0, t) = u_x(L, t)$$

then $E(t)$ is constant.

[1] This can also be thought of as saying that $E(t)$ is a conserved quantity.

Question 4. Let the function $\nu(x,t)$ describe the number of "heat particles" at a location x and time t in a metal rod.
Suppose that we divide up our rod into segments of length Δx.

Now, suppose that in an interval of time Δt, each "heat particles" moves independently at random a distance Δx to the right with probability $1/2$, and distance Δx to the left with probability $1/2$. [2] Then $\nu(x,t)$ satisfies the following equation:

$$\nu(x, t+\Delta t) = \frac{1}{2}\nu(x+\Delta x, t) + \frac{1}{2}\nu(x-\Delta x, t)$$

[3] Suppose that we can expand $\nu(x,t)$ in a Taylor series.
Show that if $\Delta x \to 0$ and $\Delta t \to 0$ in such a way that $\frac{(\Delta x)^2}{\Delta t} \to 1$, then $\nu(x,t)$ satisfies the heat equation:

$$\frac{\partial \nu}{\partial t} = \frac{1}{2}\frac{\partial^2 \nu}{\partial t^2}$$

[2] This idea that diffusion results from the random motion of individual particles is (a simplification of) one of the central ideas of Albert Einstein's 1905 paper on Brownian motion. This paper would provide some of the first convincing evidence that matter was not continuous, but instead made up of individual particles such as atoms and molecules (a very controversial idea in 1905).

[3] This formula for $\nu(x, t+\Delta t)$ above also ends up being the basis for numerically approximating solutions to the heat equation.

Answer to Question 1. (a) Plugging in $u(x,t) = X(x)T(t)$ into the heat equation,

$$\begin{aligned}
\frac{\partial}{\partial t}[u(x,t)] &= a^2 \frac{\partial^2}{\partial x^2}[u(x,t)] \\
\frac{\partial}{\partial t}[X(x)T(t)] &= a^2 \frac{\partial^2}{\partial x^2}[X(x)T(t)] \\
X(x)T'(t) &= a^2 X''(x)T(t)
\end{aligned}$$

Separating variables,

$$\frac{T'}{a^2 T} = \frac{X''}{X}$$

Since the left hand side does not depend on x, changing x cannot change either of these two expressions because they are equal to each other. Similarly, since the right hand side does not depend on t, changing t also cannot change either side of this equality. Since changing x or t does not change either side, they must be equal to the same constant, which we will call λ:

$$\frac{T'}{a^2 T} = \frac{X''}{X} = \lambda$$

(it's also OK to use $-\lambda$ instead of λ here like the textbook does, just keep in mind that this will change around the signs in some later steps) which gives us the two following ODES for $X(x)$ and $T(t)$:

$$\boxed{X'' - \lambda X = 0}$$

$$\boxed{T' - a^2 \lambda T = 0}$$

(b) If $u_x(0,t) = 0$, then this means that:

$$u_x(0,t) = X'(0)T(t) = 0 \quad \text{for all } t \geq 0$$

There are two ways for this to happen:
1. $T(t) = 0$ for all times t
2. $X'(0) = 0$

The first option is possible, but it would result in the trivial solution of $u(x,t) = 0$ for all x and t. Since we want *nontrivial* solutions to the heat equation, we will then need to set

$$\boxed{X'(0) = 0}$$

By the same argument, the other boundary condition implies

$$\boxed{X'(L) = 0}$$

(c) Here we want to find the eigenvalues and eigenfunctions of the following boundary value problem:

$$X'' - \lambda X = 0, \qquad X'(0) = X'(L) = 0$$

Here, there are three possible cases that we need to check individually:

$$\underline{\lambda > 0}, \qquad \underline{\lambda = 0}, \qquad \underline{\lambda < 0}$$

For $\underline{\lambda > 0}$, the general solution is:

$$X(x) = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}$$

Taking the derivative,

$$X'(x) = c_1\sqrt{\lambda}e^{\sqrt{\lambda}x} - c_2\sqrt{\lambda}e^{-\sqrt{\lambda}x}$$

Plugging in the first boundary condition,

$$\begin{aligned} X'(0) = c_1\sqrt{\lambda} - c_2\sqrt{\lambda} &= 0 \\ c_1 - c_2 &= 0 \\ c_1 &= c_2 \end{aligned}$$

(We can safely divide out λ here since we have assumed $\lambda > 0$ for this example, so we are not dividing by zero.)
Plugging in the second boundary condition, and using that $c_1 = c_2$,

$$\begin{aligned} X'(L) = c_1\sqrt{\lambda}e^{\sqrt{\lambda}L} - c_1\sqrt{\lambda}e^{-\sqrt{\lambda}L} &= 0 \\ c_1 e^{\sqrt{\lambda}L} - c_1 e^{-\sqrt{\lambda}L} &= 0 \\ c_1\left(e^{\sqrt{\lambda}L} - e^{-\sqrt{\lambda}L}\right) &= 0 \end{aligned}$$

Since the quantity inside the parentheses is nonzero, this means that the only solution for $\lambda > 0$ is the trivial solution.

For $\underline{\lambda = 0}$, the ODE for X is:

$$X'' = 0$$

which has general solution

$$X(x) = c_1 + c_2 x$$

Taking the derivative,

$$X'(x) = c_2$$

The boundary conditions of $X'(0) = X'(L) = 0$ then both guarantee that $c_2 = 0$, but place no restrictions on c_1, so it can be any constant.
This means that we have the eigenvalue/eigenfunction pair:

$$\lambda = 0, \qquad X_0(x) = c_0$$

Where I've renamed the constant to c_0 for reasons that will become apparent later. For $\lambda = 0$, we get the equation $T' = 0$ for T, whose solutions are also just any constant. This corresponds to the fundamental solution:

$$\boxed{u_0(x,t) = c_0}$$

For $\underline{\lambda < 0}$, we can write λ as

$$\lambda = -\omega^2$$

which lets us re-write the equation for X as:

$$X'' + \omega^2 X = 0$$

for some positive value of ω. The general solution is:

$$X(x) = c_1 \cos(\omega x) + c_2 \sin(\omega x)$$

Taking the derivative,

$$X'(x) = -c_1 \omega \sin(\omega x) + c_2 \omega \cos(\omega x)$$

Plugging in the first boundary condition,

$$\begin{aligned} X'(0) = c_2 \omega &= 0 \\ c_2 &= 0 \end{aligned}$$

since $\omega > 0$. Plugging in the second boundary condition (and using the fact that $c_2 = 0$),

$$\begin{gathered} X'(L) = -c_1 \omega \sin(\omega L) = 0 \\ c_1 = 0 \qquad \text{or} \qquad \sin(\omega L) = 0 \end{gathered}$$

If $c_1 = 0$, then both c_1 and c_2 are zero, so we just have the trivial solution $X = 0$ everywhere. Since we wwant non-trivial solutions, we then want the sine term to be zero:

$$\begin{aligned} \sin(\omega L) &= 0 \\ \omega L &= n\pi, \qquad n = 1, 2, 3, \ldots \\ \omega &= \frac{n\pi}{L}, \qquad n = 1, 2, 3, \ldots \\ \lambda = -\omega^2 &= -\frac{n^2\pi^2}{L^2}, \qquad n = 1, 2, 3, \ldots \end{aligned}$$

So the eigenvalues and corresponding eigenfunctions for $X(x)$ are:

$$\lambda_n = -\frac{n^2\pi^2}{L^2}, \qquad c_n \cos\left(\frac{n\pi x}{L}\right), \qquad n = 1, 2, 3, \ldots$$

For these values of λ, the equation for T is:

$$T' - a^2 \frac{n^2\pi^2}{L^2} T = 0$$

which has solutions

$$T = ce^{-\left(\frac{an\pi}{L}\right)^2 t}$$

Thus the fundamental solutions $u_n(x, t)$ are:

$$\boxed{u_n(x, t) = X_n(x) T_n(t) = c_n \cos\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{an\pi}{L}\right)^2 t}}$$

(d) Since the heat equation is linear, we can add solutions and multiply solutions to create more solutions. Thus the general solution is:

$$\boxed{u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = c_0 + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{an\pi}{L}\right)^2 t}}$$

(This step is like finding y_1 and y_2 for second-order equations, and then saying that the general solution $y = c_1 y_1 + c_2 y_2$)

(e) Plugging in $t = 0$ for the initial condition,

$$u(x,0) = c_0 + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi x}{L}\right) = f(x)$$

This means that we want to write $f(x)$ as an infinite sum of cosine functions.
To do this, we extend f from $[0, L]$ to $[-L, L]$ so that f is even. Then, when we take the Fourier series of this function, there will be no sine terms (because f is even) and only cosine terms.
Moreover, the cosine terms can be found from the formulas:

$$c_0 = \frac{1}{2L} \int_{-L}^{L} f(x)dx = \frac{1}{L} \int_0^L f(x)dx$$

$$c_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

Putting this all together, the solution to this heat equation problem is:

$$\boxed{u(x,t) = c_0 + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{an\pi}{L}\right)^2 t}}$$

where the coefficients are given by:

$$\boxed{c_0 = \frac{1}{L} \int_0^L f(x)dx}$$

$$\boxed{c_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx}$$

Answer to Question 2. (a) Plugging in $u(x, y, t) = X(x)Y(y)T(t)$,

$$\alpha^2(X''YT + XY''T) = XYT'$$

Separating the t components,

$$\frac{X''}{X} + \frac{Y''}{Y} = \frac{T'}{\alpha^2 T}$$

Since the left hand side depends only on x and y, while the right hand side depends only on t, both sides must be equal to the same constant, which we will call $-\lambda^2$ (in order to match the given answers). That means

$$\frac{X''}{X} + \frac{Y''}{Y} = \frac{T'}{\alpha^2 T} = -\lambda^2$$

which gives us the ODE for $T(t)$:

$$\boxed{T' + \alpha^2\lambda^2 T}$$

this leaves us with

$$\frac{X''}{X} + \frac{Y''}{Y} = -\lambda^2$$

Separating variables again,

$$\frac{X''}{X} = -\lambda^2 - \frac{Y''}{Y}$$

Since the left hand side depends only on x and the right hand side depends only on y, both sides must be equal to the same constant, which we will call $-\mu^2$ (in order to match the given answers).

$$\frac{X''}{X} = -\lambda^2 - \frac{Y''}{Y} = -\mu^2$$

Rearranging these equations gives us the ODEs for $X(x)$ and $Y(y)$:

$$\boxed{X'' + \mu^2 X = 0}$$

$$\boxed{Y'' + (\lambda^2 - \mu^2)Y = 0}$$

(Note: there are many different acceptable answers here in terms of whether to use λ, $-\lambda$, or $-\lambda^2$ etc. The ODEs will look slightly different, but the end results will be the same).

(b) Plugging in $u(r, \theta, t) = R(r)\Theta(\theta)T(t)$, and then separating variables

$$\alpha^2\left(R''\Theta T + \frac{R'\Theta T}{r} + \frac{R\Theta'' T}{r^2}\right) = R\Theta T'$$

$$\frac{R''}{R} + \frac{R'}{rR} + \frac{\Theta''}{r^2\Theta} = \frac{T'}{\alpha^2 T}$$

Since the left hand side depends only on r and θ, while the right hand side depends only on t, both sides must equal the same constant $-\lambda^2$,

$$\frac{R''}{R} + \frac{R'}{rR} + \frac{\Theta''}{r^2\Theta} = \frac{T'}{\alpha^2 T} = -\lambda^2$$

This gives us the ODE for $T(t)$:

$$\boxed{T' + \alpha^2\lambda^2 T = 0}$$

Leaving us with

$$\frac{R''}{R} + \frac{R'}{rR} + \frac{\Theta''}{r^2\Theta} = -\lambda^2$$

Separating the r and θ components,

$$\frac{R''}{R} + \frac{R'}{rR} + \lambda^2 = \frac{-\Theta''}{r^2\Theta}$$
$$\frac{r^2R''}{R} + \frac{rR'}{R} + \lambda^2 r^2 = \frac{-\Theta''}{\Theta} = \mu^2$$

for some constant μ. This gives us the ODE for $\Theta(\theta)$:

$$\boxed{\Theta'' + \mu^2\Theta = 0}$$

and the ODE for $R(r)$:

$$\boxed{r^2R'' + rR'' + (\lambda^2 r^2 - \mu^2)R = 0}$$

Answer to Question 3. (a) If we take the derivative of E, then we get the derivative (with respect to t) of an integral (with respect to x):

$$\frac{dE}{dt} = \frac{d}{dt}\int_0^L u(x,t)dx$$

Since this derivative and integral are with respect to different variables x and t, we can actually interchange them (a more general form of this is known as Leibniz's formula):

$$\frac{dE}{dt} = \int_0^L \frac{\partial}{\partial t}u(x,t)dx$$

Since u is a solution to the heat equation $u_t = \alpha^2 u_x x$, we can then replace the u_t in our integrand:

$$\frac{dE}{dt} = \alpha^2\int_0^L \frac{\partial^2}{\partial x^2}u(x,t)dx$$

By the fundamental theorem of calculus, we can integrate $u_x x$ by just evaluating u_x at the endpoints of the integral:

$$\frac{dE}{dt} = \alpha^2\left[u_x(x,t)\right]_0^L = \alpha^2 u_x(L,t) - \alpha^2 u_x(0,t)$$

If the metal rod is insulated, then both u_x terms are zero:

$$\frac{dE}{dt} = 0 - 0 = 0$$

and therefore $E(t)$ is constant.

(b) By the same argument as part **(a)** ,

$$\frac{dE}{dt} = \alpha^2 u_x(L,t) - \alpha^2 u_x(0,t)$$

and if the ends are connected, both of these terms cancel out, leaving:

$$\frac{dE}{dt} = 0 \qquad \Longrightarrow \qquad E(t) \text{ is constant}$$

Answer to Question 4. If we expand $v(x, t + \Delta t)$ as a Taylor series, we get:

$$v(x, t+\Delta t) = v + \frac{\partial v}{\partial t}\Delta t + \mathcal{O}(\Delta t^2)$$

(in case you haven't seen it before, $\mathcal{O}$ is what's called big-O notation, and it means terms of order Δt^2 or greater, since they will end up disappearing in the limit.)
Similarly, expanding $v(x \pm \Delta x, t)$, we get:

$$\begin{aligned} v(x+\Delta x) &= v + \frac{\partial v}{\partial x}\Delta x + \frac{1}{2}\frac{\partial^2 v}{\partial x^2}(\Delta x)^2 + \mathcal{O}(\Delta x^3) \\ v(x-\Delta x) &= v - \frac{\partial v}{\partial x}\Delta x + \frac{1}{2}\frac{\partial^2 v}{\partial x^2}(\Delta x)^2 + \mathcal{O}(\Delta x^3) \end{aligned}$$

Plugging this all into the formula above and cancelling things out,

$$\begin{aligned} v(x, t+\Delta t) &= \frac{1}{2}v(x+\Delta x, t) + \frac{1}{2}v(x-\Delta x, t) \\ v + \frac{\partial v}{\partial t}\Delta t + \mathcal{O}(\Delta t^2) &= \frac{1}{2}\left[v + \frac{\partial v}{\partial x}\Delta x + \frac{1}{2}\frac{\partial^2 v}{\partial x^2}(\Delta x)^2 + v - \frac{\partial v}{\partial x}\Delta x + \frac{1}{2}\frac{\partial^2 v}{\partial x^2}(\Delta x)^2\right] + \mathcal{O}(\Delta x^3) \\ v + \frac{\partial v}{\partial t}\Delta t + \mathcal{O}(\Delta t^2) &= v + \frac{1}{2}\frac{\partial^2 v}{\partial x^2}(\Delta x)^2 + \mathcal{O}(\Delta x^3) \\ \frac{\partial v}{\partial t}\Delta t + \mathcal{O}(\Delta t^2) &= \frac{1}{2}\frac{\partial^2 v}{\partial x^2}(\Delta x)^2 + \mathcal{O}(\Delta x^3) \\ \frac{\partial v}{\partial t} + \mathcal{O}(\Delta t) &= \frac{1}{2}\frac{\partial^2 v}{\partial x^2}\frac{(\Delta x)^2}{\Delta t} + \mathcal{O}\left(\frac{\Delta x^3}{\Delta t}\right) \end{aligned}$$

Then, taking the limit as $\Delta t \to 0$ and $\frac{(\Delta x)^2}{\Delta t} \to 1$, we are left with:

$$\boxed{\frac{\partial v}{\partial t} = \frac{1}{2}\frac{\partial^2 v}{\partial x^2}}$$

so $v(x, t)$ is a solution to this heat equation.