Math 2930 Worksheet

BVPs, Fourier Series

Week 11

November 2nd, 2017

Question 1. (*) Consider the function $f(x)$ defined by:

$$\begin{aligned} f(x) &= -1, \qquad -2 \leq x < 0 \\ f(x) &= 0, \qquad\quad 0 \leq x < 2 \\ f(x+4) &= f(x) \end{aligned}$$

(a) Sketch $f(x)$ for several periods

(b) *Without* doing any calculations, to what value does the Fourier series of $f(x)$ converge at $x = \pi$?

(c) Find the Fourier series for $f(x)$

Question 2. (*) Solve the two-point boundary value problem

$$y'' + 3y = \cos(x), \qquad y'(0) = 0, \qquad y'(\pi) = 0$$

Question 3. (*) Solve the eigenvalue problem

$$y'' + \lambda y = 0$$

subject to the constraints $y(0) = y(\pi) = y(2\pi/3) = 0, \quad \lambda > 0.$

Question 4. In solving certain PDE problems using separation of variables, you need to expand a given function $f(x)$ defined on $[0, L]$ as a sum of sine functions with *odd* indices only:

$$\sin\left(\frac{\pi x}{2L}\right), \sin\left(\frac{3\pi x}{2L}\right), \sin\left(\frac{5\pi x}{2L}\right), \ldots$$

This question is meant to help walk you through that process.

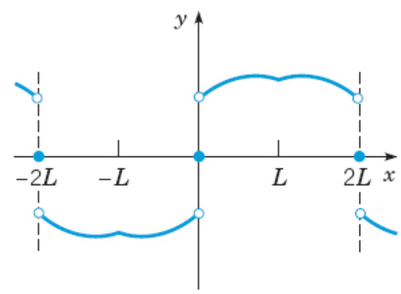
(a) To do this, f should first be extended into $(L, 2L)$ so that it is symmetric about $x = L$. Let the resulting function be extended into $(-2L, 0)$ as an odd function and elsewhere as a periodic function of period $4L$ (see picture below).

Show that this new function has a Fourier series in terms of

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{(2n-1)\pi x}{2L}\right)$$

where

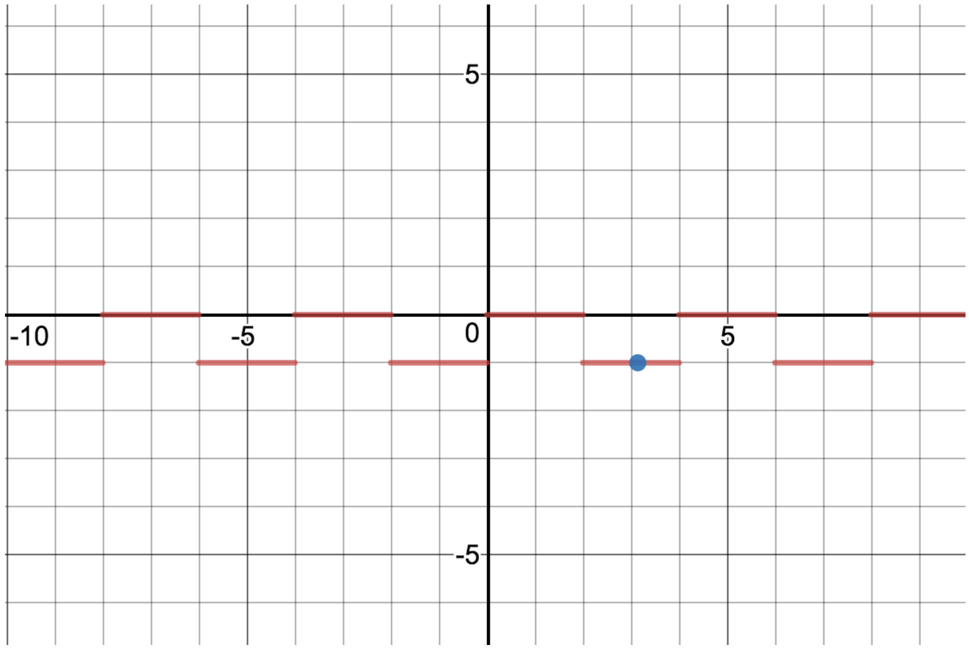
$$b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{(2n-1)\pi x}{2L}\right) dx$$



(b) How should a function f defined on $[0, L]$ be extended so as to obtain a Fourier series involving only the functions:

$$\cos\left(\frac{\pi x}{2L}\right), \cos\left(\frac{3\pi x}{2L}\right), \cos\left(\frac{5\pi x}{2L}\right), \ldots?$$

Answer to Question 1. (a) Here is a graph of $f(x)$:



(b) Since $f(x)$ is continuous at $x = \pi$ (denoted by the blue dot in the picture above), the Fourier Series for f converges to the value $f(x)$, which is just -1.
In terms of equations,

$$\boxed{\lim_{N\to\infty}\left[\frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos\left(\frac{n\pi^2}{2}\right) + b_n \sin\left(\frac{n\pi^2}{2}\right)\right] = -1}$$

(*Note*: just the first sentence here would be a sufficient answer on a quiz/exam)

(c) For this problem, $L = 2$. Using the formulas for the Fourier series coefficients:

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx = \frac{1}{2}\int_{-2}^{0} -1dx + \frac{1}{2}\int_{0}^{2} 0dx = -1$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{n\pi x}{2}\right)dx = \frac{1}{2}\int_{-2}^{0} -\cos\left(\frac{n\pi x}{2}\right)dx$$

$$a_n = \frac{-1}{2}\left[\frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\right]_{-2}^{0} = \frac{-1}{n\pi}[\sin(0) - \sin(n\pi)] = 0$$

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{n\pi x}{2}\right)dx = \frac{1}{2}\int_{-2}^{0} -\sin\left(\frac{n\pi x}{2}\right)dx$$

$$b_n = \frac{1}{2}\left[\frac{2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\right]_{-2}^{0} = \frac{1}{n\pi}\left[\cos(0) - \cos(-n\pi)\right]$$

$$b_n = \frac{1-(-1)^n}{n\pi} = \begin{cases} \frac{2}{n\pi}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

So this Fourier series can be written as:

$$\boxed{f(x) = \frac{-1}{2} + \sum_{n \text{ odd}} \frac{2}{n\pi} \sin\left(\frac{n\pi x}{2}\right) = \frac{-1}{2} + \sum_{n=1}^{\infty} \frac{2}{(2n-1)\pi} \sin\left(\frac{(2n-1)\pi x}{2}\right)}$$

Answer to Question 2. The homogeneous equation is just

$$y'' + 3y = 0$$

solving the characteristic polynomial for r,

$$\begin{aligned} r^2 + 3 &= 0 \\ r^2 &= -3 \\ r &= \pm\sqrt{3}i \end{aligned}$$

The corresponding homogeneous solution is:

$$y_h(x) = c_1 \cos\left(\sqrt{3}x\right) + c_2 \sin\left(\sqrt{3}x\right)$$

For the particular solution, we guess a solution of the form

$$Y(x) = A\cos(x)$$

(we could also add a $B\sin(x)$ term, but we won't need one because there are only even-order derivatives in this problem.)
Taking derivatives,

$$\begin{aligned} Y(x) &= A\cos(x) \\ Y'(x) &= -A\sin(x) \\ Y''(x) &= -A\cos(x) \end{aligned}$$

and plugging this into the original equation,

$$\begin{aligned} Y'' + 3Y &= \cos(x) \\ (-A\cos(x)) + 3(A\cos(x)) &= \cos(x) \\ 2A\cos(x) &= \cos(x) \\ A &= \frac{1}{2} \end{aligned}$$

So the particular solution is

$$Y(x) = \frac{1}{2}\cos(x)$$

Combining the homogeneous and particular solutions, we get the general solution:

$$y(x) = y_h(x) + Y(x) = c_1 \cos\left(\sqrt{3}x\right) + c_2 \sin\left(\sqrt{3}x\right) + \frac{1}{2}\cos(x)$$

Taking the derivative,

$$y'(x) = -\sqrt{3}c_1 \sin\left(\sqrt{3}x\right) + \sqrt{3}c_2 \cos\left(\sqrt{3}x\right) - \frac{1}{2}\sin(x)$$

Now that we have the general solution, we can plug in the boundary conditions:

$$\begin{aligned} y'(0) &= 0 + \sqrt{3}c_2(1) - 0 = \sqrt{3}c_2 = 0 \\ c_2 &= 0 \\ y'(\pi) &= -\sqrt{3}c_1 \sin\left(\sqrt{3}\pi\right) - \frac{1}{2}\sin(\pi) \end{aligned}$$

Since $\sin(\sqrt{3}\pi) \neq 0$, this means that $c_1 = 0$ as well.
Therefore the solution to the boundary value problem is just the particular solution:

$$\boxed{y(x) = \frac{1}{2}\cos(x)}$$

Answer to Question 3. The general solution to the equation is:

$$y(x) = c_1 \cos\left(\sqrt{\lambda}x\right) + c_2 \sin\left(\sqrt{\lambda}x\right)$$

Plugging in the first boundary condition of $y(0) = 0$,

$$y(0) = c_1 \cos(0) + c_2 \sin(0) = c_1 = 0$$

so the solution looks like

$$y(x) = c_2 \sin\left(\sqrt{\lambda}x\right)$$

Plugging in the second and third boundary condition of $y(2\pi/3) = y(\pi) = 0$,

$$\begin{aligned} y(2\pi/3) &= c_2 \sin\left(\sqrt{\lambda}\frac{2\pi}{3}\right) = 0 \\ y(\pi) &= c_2 \sin\left(\sqrt{\lambda}\pi\right) = 0 \end{aligned}$$

If $c_2 = 0$, then this is just the trivial solution.
So in order to have nontrivial solutions, we need that both

$$\sin\left(\sqrt{\lambda}\frac{2\pi}{3}\right), \qquad \sin\left(\sqrt{\lambda}\pi\right)$$

are *simultaneously* zero. This happens when

$$\frac{2\sqrt{\lambda}}{3} \text{ is an integer, } \textit{and } \sqrt{\lambda} \text{ is an integer}$$

(but not necessarily the same integer)

This happens if and only if $\sqrt{\lambda}$ is a multiple of 3, *i.e.*

$$\sqrt{\lambda} = 3n, \quad n = 1, 2, 3, \ldots$$

Squaring both sides, we get that the eigenvalues are:

$$\lambda_n = 9n^2, \quad n = 1, 2, 3, \ldots$$

with the corresponding eigenfunctions

$$y_n = c_2 \sin(3nx)$$

where c_2 can be any constant.

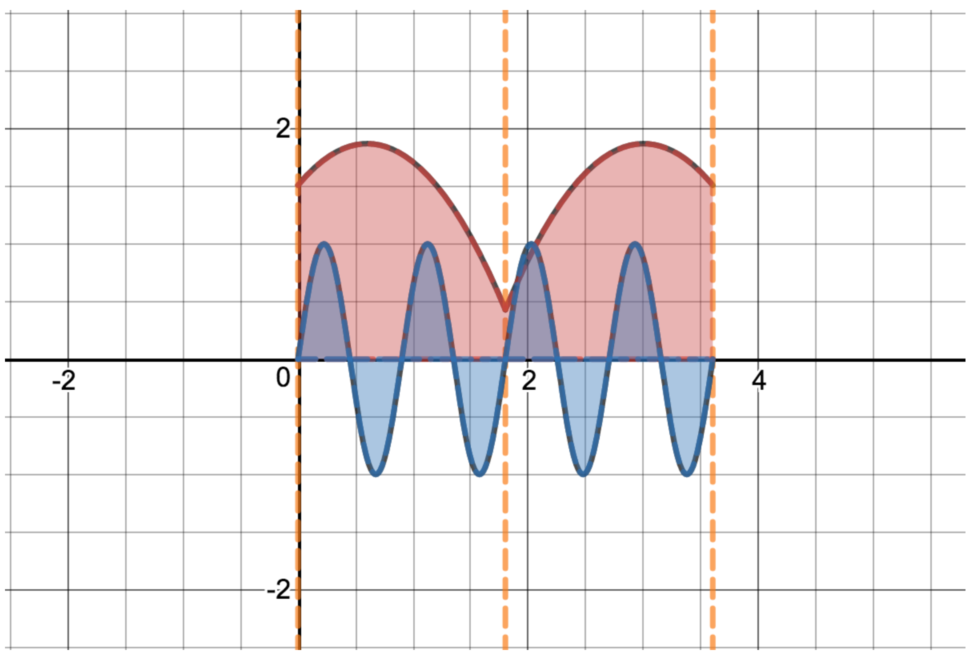
Answer to Question 4. (a) Since the given extension of the function is even, we know that $a_n = 0$ for every n, and the function can be written as a sum of sine functions only.
Calculating the coefficients of these sine terms (using the fact that f is odd):

$$\begin{aligned} b_n &= \frac{1}{2L}\int_{-2L}^{2L} f(x)\sin\left(\frac{n\pi x}{2L}\right)dx \\ &= \frac{2}{2L}\int_{0}^{2L} f(x)\sin\left(\frac{n\pi x}{2L}\right)dx \\ &= \frac{1}{L}\int_{0}^{2L} f(x)\sin\left(\frac{n\pi x}{2L}\right)dx \end{aligned}$$

We can take advantage of the additional symmetry in this problem by breaking down this integral into two pieces, one from 0 to L and the other from L to $2L$ as follows:

$$b_n = \frac{1}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{2L}\right)dx + \frac{1}{L}\int_{L}^{2L} f(x)\sin\left(\frac{n\pi x}{2L}\right)dx$$

Now, for even values of n, the way we have extended f so that it is symmetric about L makes sure that these two integrals cancels out. (See the picture below)

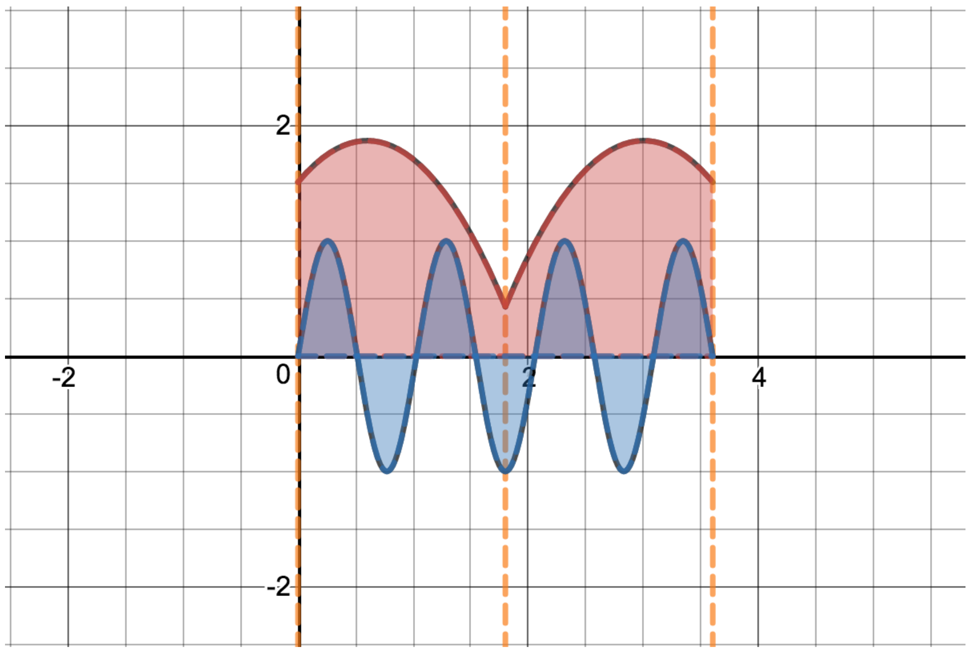


The (even-indexed) sine function is in blue and the extension of f is in red. Looking at this picture, you can see that sin is odd about $x = L$, and f is even about $x = L$, so b_n will be the integral from 0 to 2L of a function that is odd about $x = L$.
In other words, the integral from 0 to L and the integral from L to 2L above will cancel each other out:

$$b_n = \frac{1}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{2L}\right)dx + \frac{1}{L}\int_L^{2L} f(x)\sin\left(\frac{n\pi x}{2L}\right)dx = 0$$

Thus $b_n = 0$ when n is even.

Now, if n is odd, the picture looks something like:



Now we see that everything is perfectly symmetric about $x = L$, so these two integrals will equal the same value:

$$\begin{aligned} b_n &= \frac{1}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{2L}\right)dx + \frac{1}{L}\int_L^{2L} f(x)\sin\left(\frac{n\pi x}{2L}\right)dx \\ &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{2L}\right)dx \end{aligned}$$

So the b_n coefficients are given by:

$$b_n = \begin{cases} 0, & n \text{ even} \\ \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{2L}\right)dx, & n \text{ odd} \end{cases}$$

Putting this all together, the Fourier series expansion of $f(x)$ is:

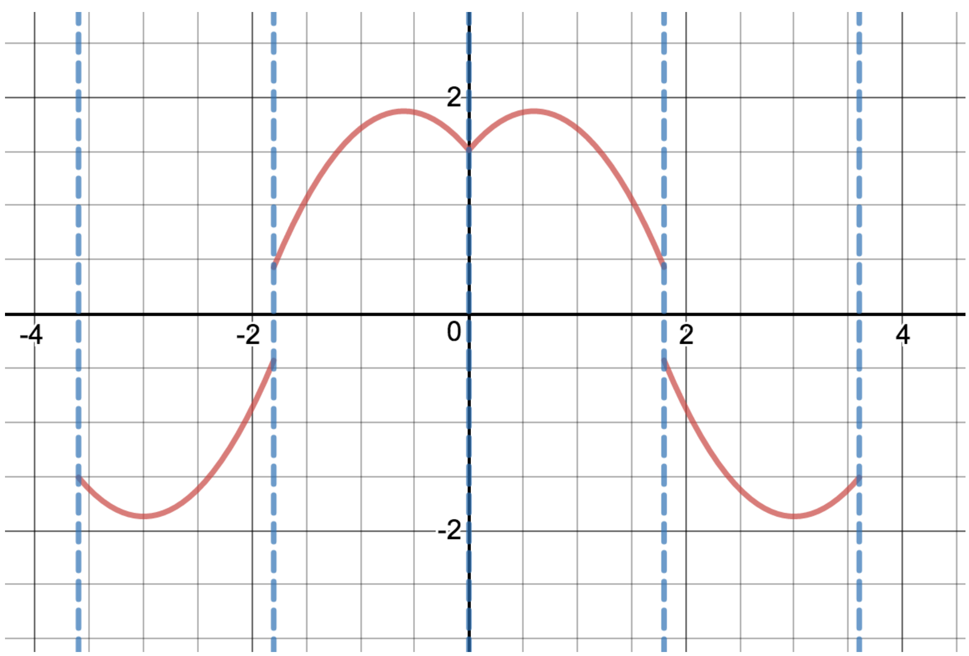
$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2L}\right) \\ &= \sum_{n \text{ odd}} \left[\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{2L}\right)dx\right]\sin\left(\frac{n\pi x}{2L}\right) \\ &= \sum_{n=1}^{\infty} \left[\frac{2}{L}\int_0^L f(x)\sin\left(\frac{(2n-1)\pi x}{2L}\right)dx\right]\sin\left(\frac{(2n-1)\pi x}{2L}\right) \end{aligned}$$

so it only has the $\sin\left(\frac{(2n-1)\pi x}{2L}\right)$ terms as desired.

(b) To get only the odd-indexed cosine functions in our Fourier series, we can repeat a similar extension to the one in part **(a)** .
We can extend $f(x)$ from $[0, L]$ to $[-2L, 2L]$ as follows:

- Extend f to $[0, 2L]$ so that it is odd around $x = L$
- Extend this to $[-2L, 2L]$ so that it is even around $x = 0$

The resulting extension is illustrated by the following image:



Repeating similar arguments from **(a)** , but with even and odd switched, we can get that this results in a Fourier series with only the desired cosine terms.