Math 2930 Worksheet
Prelim 2 Review

Week 10
October 26th, 2017

Question 1. (*) Determine the general solution of

$$y'' + \lambda^2 y = \sum_{m=1}^{N} a_m \sin(\pi m t)$$

where $\lambda > 0$ and $\lambda \neq m\pi$ for $m = 1, ..., N$

Question 2. (*) For the ODE:

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + \lambda y = 0$$

(a) Find the general solution for $\lambda > 0$

(b) *(Just to be clear: this part would not be fair game for Prelim 2 this semester, but we should have covered everything you need to solve it in class already.)*

Find the eigenvalues λ which satisfy the following BC:

$$y(1) = 0 \qquad \text{and} \qquad y(2) = 0$$

Question 3. (*) Find the general solution of the differential equation

$$y'' - y' + \frac{1}{4}y = 3 + e^{t/2}$$

Question 4. (*) A mass of 1 kg stretches a spring 8 cm. The mass is first pushed upward, contracting the spring a distance of 2 cm, and then set in motion with a downward velocity of 60 cm/s. Assuming that there is no damping and no external force is applied,

(a) Find the position $u(t)$ of the mass at any time t

(b) Determine the period, frequency and the amplitude of the motion

Question 5. (*) Using the method of variation of parameters, find the general solution of the differential equation

$$y'' + 4y' + 4y = t^{-2}e^{-2t}, \qquad t > 0$$

(if this were on the prelim, you would also be given something like the following formulas)

Variation of Parameters: $Ly = g, \qquad y = u_1y_1 + u_2y_2$

$$u_1 = -\int \frac{y_2 g}{W}dt, \qquad u_2 = +\int \frac{y_1 g}{W}dt, \qquad \text{where } W = y_1y_2' - y_2y_1'$$

Question 6. (*) Find the solution of the initial value problem

$$y''' - y'' + y' - y = 0, \qquad y(0) = 2, \qquad y'(0) = -1. \qquad y''(0) = -2$$

Answer to Question 1. For the homogenous solution, we write down the characteristic equation, and solve for r:

$$\begin{aligned} r^2 + \lambda^2 &= 0 \\ r^2 &= -\lambda^2 \\ r &= \pm\lambda i \end{aligned}$$

So the homogenous solution is:

$$y_h(t) = c_1 \cos(\lambda t) + c_2 \sin(\lambda t)$$

Now, since our equation is linear, we can find a particular solution for each term in the right hand side individually, and then add them together. In other words, if $Y_m(t)$ is a particular solution of

$$y'' + \lambda^2 y = a_m \sin(\pi m t)$$

then $Y(t) = \sum_{m=1}^{n} Y_m(t)$ is a particular solution of:

$$y'' + \lambda^2 y = \sum_{m=1}^{N} a_m \sin(\pi m t)$$

So we'll just find a particular solution Y_m for $a_m \sin(m\pi t)$ first.
To do that, we guess a Y_m of the form:

$$Y_m(t) = B_m \cos(m\pi t) + C_m \sin(m\pi t)$$

Taking derivatives,

$$\begin{aligned} Y_m'(t) &= -m\pi B_m \sin(m\pi t) + m\pi C_m \cos(m\pi t) \\ Y_m''(t) &= -m^2\pi^2 B_m \cos(m\pi t) - m^2\pi^2 C_m \sin(m\pi t) \end{aligned}$$

So plugging our guess for Y_m into the original equation,

$$\begin{aligned} Y_m'' + \lambda^2 Y_m &= a_m \sin(m\pi t) \\ -m^2\pi^2 B_m \cos(m\pi t) - m^2\pi^2 C_m \sin(m\pi t) + \lambda^2 B_m \cos(m\pi t) + \lambda^2 C_m \sin(m\pi t) &= a_m \sin(m\pi t) \\ (\lambda^2 - m^2\pi^2) B_m \cos(m\pi t) + (\lambda^2 - m^2\pi^2) C_m \sin(m\pi t) &= a_m \sin(m\pi t) \end{aligned}$$

So that gives us two equations for B_m and C_m:

$$\begin{aligned} (\lambda^2 - m^2\pi^2) B_m &= 0 \\ (\lambda^2 - m^2\pi^2) C_m &= a_m \end{aligned}$$

The solution to this system is:

$$B_m = 0, \qquad C_m = \frac{a_m}{\lambda^2 - m^2\pi^2}$$

So our particular solution Y_m is:

$$Y_m(t) = \frac{a_m}{\lambda^2 - m^2\pi^2} \sin(m\pi t)$$

Adding these all together, our whole particular solution $Y(t)$ is:

$$Y(t) = \sum_{m=1}^{N} y_m(t) = \sum_{m=1}^{N} \frac{a_m}{\lambda^2 - m^2\pi^2} \sin(m\pi t)$$

Then we add the homogenous and particular solutions together to get the general solution:

$$\boxed{y(t) = y_h(t) + Y(t) = c_1 \cos(\lambda t) + c_2 \sin(\lambda t) + \sum_{m=1}^{N} y_m(t) = \sum_{m=1}^{N} \frac{a_m}{\lambda^2 - m^2\pi^2} \sin(m\pi t)}$$

Answer to Question 2. (a) Since this is an Euler equation, we guess an answer of the form $y = x^r$ for some unknown power r. Plugging this in,

$$\begin{aligned} x^2 r(r-1)x^{r-2} + xrx^{r-1} + \lambda x^r &= 0 \\ r(r-1)x^r + rx^r + \lambda x^r &= 0 \\ \big[r(r-1) + r + \lambda\big]x^r &= 0 \end{aligned}$$

So our characteristic equation is:

$$r^2 + \lambda = 0$$

Since $\lambda > 0$, the roots are imaginary:

$$r = \pm\sqrt{\lambda} i$$

This corresponds to a general solution of:

$$\boxed{y(x) = c_1 \cos\left(\sqrt{\lambda}\ln(x)\right) + c_2 \sin\left(\sqrt{\lambda}\ln(x)\right)}$$

(b) Plugging in the first boundary condition,

$$\begin{aligned} y(1) = c_1 \cos\left(\sqrt{\lambda}\ln(1)\right) + c_2 \sin\left(\sqrt{\lambda}\ln(1)\right) &= 0 \\ c_1 \cos(0) + c_2 \sin(0) &= 0 \\ c_1 &= 0 \end{aligned}$$

Plugging in the second boundary condition,

$$\begin{aligned} y(2) = c_1 \cos\left(\sqrt{\lambda}\ln(2)\right) + c_2 \sin\left(\sqrt{\lambda}\ln(2)\right) &= 0 \\ c_2 \sin\left(\sqrt{\lambda}\ln(2)\right) &= 0 \end{aligned}$$

One way of satisfying this boundary condition is to set $c_2 = 0$, but then we're just left with the trivial solution $y = 0$.

However, we can have nontrivial solutions if:

$$\sin\left(\sqrt{\lambda}\ln(2)\right) = 0$$
$$\sqrt{\lambda}\ln(2) = n\pi, \qquad n = 1, 2, 3, ...$$
$$\lambda = \frac{n^2}{\ln(2)^2}, \qquad n = 1, 2, 3, ...$$

So the eigenvalues are:

$$\lambda_n = \frac{n^2}{\ln(2)^2}, \qquad n = 1, 2, 3, ...$$

with corresponding eigenfunctions:

$$y_n(x) = c_2 \sin\left(\sqrt{\lambda}\ln(x)\right)$$
$$y_n(x) = c_2 \sin\left(\frac{n\ln(x)}{\ln(2)}\right)$$

where c_2 can be any constant

Answer to Question 3. First the solution to the homogenous equation.

$$r^2 - r + \frac{1}{4} = 0$$
$$\left(r - \frac{1}{2}\right)^2 = 0$$
$$r = \frac{1}{2}, \quad \frac{1}{2} \quad \text{(repeated)}$$

So the homogenous part of the solution is:

$$y_h(t) = c_1 e^{t/2} + c_2 t e^{t/2}$$

Since $e^{t/2}$ shows up twice in our homogenous solution, we guess a particular solution of the form:

$$Y(t) = At^2 e^{t/2} + B$$

taking derivatives,

$$Y'(t) = \left(\frac{A}{2}t^2 + 2At\right) e^{t/2}$$
$$Y''(t) = \left(\frac{A}{4}t^2 + 2At + 2A\right) e^{t/2}$$

Plugging this into the original equation,

$$Y'' - Y' + \frac{1}{4}Y = \left[\left(\frac{A}{4}t^2 + 2At + 2A\right) - \left(\frac{A}{2}t^2 + 2At\right) + \frac{1}{4}(At^2)\right] e^{t/2} + \frac{B}{4} = 3 + e^{t/2}$$
$$\left[0t^2 + 0t + 2A\right] e^{t/2} + \frac{B}{4} = 3 + e^{t/2}$$
$$2Ae^{t/2} + \frac{B}{4} = 3 + e^{t/2}$$
$$2A = 1, \qquad \frac{B}{4} = 3$$
$$A = \frac{1}{2}, \qquad B = 12$$

So the particular solution is:

$$Y(t) = \frac{1}{2}t^2 e^{t/2} + 12$$

and the general solution is:

$$\boxed{y(t) = y_h(t) + Y(t) = c_1 e^{t/2} + c_2 t e^{t/2} + \frac{1}{2}t^2 e^{t/2} + 12}$$

Answer to Question 4. *Note: I will do this problem entirely in meters. It's possible to do this all in centimeters, but you then have to remember to convert the gravitational constant* g *into centimeters per second squared. Also, I'll use* $g = 10\mathrm{m/s}$*, but* $g = 9.8\mathrm{m/s}$ *is certainly acceptable as well, it will just result in slightly different answers.*

(a) To find the spring constant, we'll use the information in the first sentence. A mass of 1 kg stretching a spring 8cm means that the weight of the mass is equal to the spring force:

$$\begin{aligned} mg &= kx \\ (1\mathrm{kg})(10\mathrm{m/s^2}) &= k(0.08\mathrm{m}) \\ k &= \frac{10\mathrm{m/s^2}}{0.08\mathrm{m}} = 125\mathrm{N/m} \end{aligned}$$

Since there is no damping, and no external force, the position $u(t)$ of the mass satisfies the following equation:

$$mu'' + ku = 0$$

Solving the characteristic equation,

$$\begin{aligned} mr^2 + k &= 0 \\ r^2 &= -\frac{k}{m} \\ r &= \pm\sqrt{\frac{k}{m}}i \end{aligned}$$

So the general solution is:

$$u(x) = c_1 \cos\left(\sqrt{\frac{k}{m}}t\right) + c_2 \sin\left(\sqrt{\frac{k}{m}}t\right)$$

Plugging in the initial conditions (*converted to meters*),

$$\begin{aligned} u(0) &= c_1 = 0.02 \\ u'(0) &= c_2\sqrt{\frac{k}{m}} = -0.6 \end{aligned}$$

So the coefficients are:

$$c_1 = 0.02, \qquad c_2 = -0.6\sqrt{\frac{m}{k}}$$

giving a solution of

$$u(x) = 0.02\cos\left(\sqrt{\frac{k}{m}}t\right) - 0.6\sqrt{\frac{m}{k}}\sin\left(\sqrt{\frac{k}{m}}t\right)$$

Plugging in the values of $m = 1\mathrm{kg}$ and $k = 125\mathrm{N/m}$, the solution is:

$$\boxed{u(x) = 0.02\cos\left(5\sqrt{5}t\right) - \frac{0.6}{5\sqrt{5}}\sin\left(5\sqrt{5}t\right)}$$

(b) From the answer to part **(a)** , we see that the radial frequency is:

$$\boxed{\omega = 5\sqrt{5}\mathrm{rad/s}}$$

the period is

$$\boxed{T = \frac{2\pi}{\omega} = \frac{2\pi}{5\sqrt{5}}\text{s}}$$

and the amplitude is

$$\boxed{R = \sqrt{(0.02)^2 + \left(\frac{-0.6}{5\sqrt{5}}\right)^2} = \frac{0.02\sqrt{41}}{5}\text{m}}$$

Answer to Question 5. First, we solve the homogenous equation:

$$\begin{aligned} y'' + 4y' + 4y &= 0 \\ r^2 + 4r + 4 &= 0 \\ (r+2)^2 &= 0 \\ r &= -2, \quad -2 \quad \text{(repeated)} \end{aligned}$$

So the homogenous solution is given by:

$$y_1(t) = e^{-2t}, \qquad y_2(t) = te^{-2t}$$

We calculate the Wronskian:

$$\begin{aligned} W[y_1, y_2](t) &= y_1 y_2' - y_2 y_1' \\ &= e^{-2t}\left(e^{-2t} - 2te^{-2t}\right) + 2e^{-2t}\left(te^{-2t}\right) \\ &= e^{-4t} \end{aligned}$$

Now we can calculate u_1 and u_2 using our formulas for variation of parameters:

$$\begin{aligned} u_1 &= -\int \frac{te^{-2t}t^{-2}e^{-2t}}{e^{-4t}}dt = -\int \frac{1}{t}dt = -\ln(t) + C_1 \\ u_2 &= \int \frac{e^{-2t}t^{-2}e^{-2t}}{e^{-4t}}dt = \int \frac{1}{t^2}dt = \frac{-1}{t} + C_2 \end{aligned}$$

So our general solution is:

$$\begin{aligned} y(t) &= u_1(t)y_1(t) + u_2(t)y_2(t) \\ y(t) &= (-\ln(t) + C_1)\, e^{-2t} + \left(\frac{-1}{t} + C_2\right) te^{-2t} \end{aligned}$$

which simplifies to:

$$\boxed{y(t) = c_1 e^{-2t} + c_2 te^{-2t} - \ln(t)e^{-2t}}$$

Answer to Question 6. To find the general solution, we find the roots of the characteristic equation:

$$\begin{aligned} y''' - y'' + y' - y &= 0 \\ r^3 - r^2 + r - 1 &= 0 \\ (r^2+1)(r-1) &= 0 \\ r &= \pm i, \quad 1 \end{aligned}$$

so the corresponding general solution is:

$$y(t) = c_1 \cos(t) + c_2 \sin(t) + c_3 e^t$$

Plugging in the initial conditions, we get a system of equations for c_1, c_2, c_3:

$$\begin{aligned} y(0) &= c_1 + c_3 = 2 \\ y'(0) &= c_2 + c_3 = -1 \\ y''(0) &= -c_1 + c_3 = -2 \end{aligned}$$

The solution to this system of equations is

$$c_1 = 2, \qquad c_2 = -1, \qquad c_3 = 0$$

So the solution to the initial value problem is then

$$\boxed{y(t) = 2\cos(t) - \sin(t)}$$