Math 2930 Worksheet

Introduction to Differential Equations

Week 1

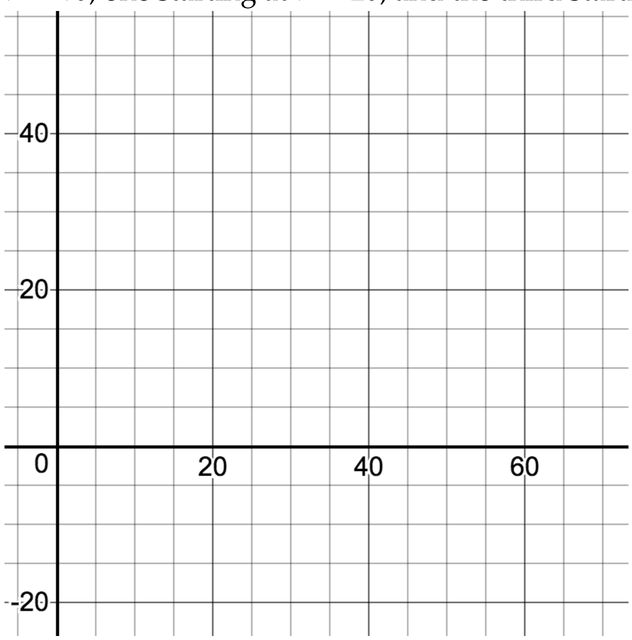
August 24, 2017

Question 1. Is the function $y = 1 + t$ a solution to the differential equation $\frac{dy}{dt} = \frac{y^2-1}{t^2+2t}$?
How about the function $y = 1 + 2t$?
How about $y = 1$?

Question 2. Differential equations are often used to model population growth, say of the number of fish in a lake as a function of time. Let's simplify the situation by making the following assumptions:

- There is only one species (*e.g.* fish)
- The species has been in its habitat (*e.g.* a lake) for some time prior to what we call $t = 0$
- The species has access to unlimited resources (*e.g.* food, space, water)
- The species reproduces continuously

Given these assumptions, sketch three possible graphs of population versus time: one starting at $P = 10$, one starting at $P = 20$, and the third starting at $P = 30$.



(a) For your graph starting with $P = 10$, how does the slope vary as time increases?

(b) For a set P value, say $P = 30$, how do the slopes vary across the three graphs you drew?

Question 3. The situation in Question 2 can be modeled with a differential equation of the form $\frac{dP}{dt} = something$. Here are some possible equations someone might use to try and model this situation. For each one, come up with reasons for why they do or don't accurately model the problem.

- $\frac{dP}{dt} = (t+1)^2$

- $\frac{dP}{dt} = 2P$

- $\frac{dP}{dt} = Ce^t$

- $\frac{dP}{dt} = 2t$

- $\frac{dP}{dt} = P^2 + 1$

Question 4. Below are two systems of differential equations. In both of these systems, x and y refer to the population of two different species at time t. Which system describes a situation where the two species compete? Which system describes cooperative species? Explain your reasoning.

$$(i) \quad \frac{dx}{dt} = -5x + 2xy, \qquad \frac{dy}{dt} = -4y + 3xy,$$

$$(ii) \quad \frac{dx}{dt} = x - 2xy \qquad \frac{dy}{dt} = 2y - xy$$

Question 5. The differential equation

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y$$

where r and K are positive constants, is an example of what is known as a *logistic equation*. These are usually used to model the population y of some species in an environment with limited resources as a function of time t.

(a) If we start with some initial population $y(0) > 0$, what can you say about the qualitative behavior of $y(t)$ will be as t increases? By qualitative I mean try to explain using words instead of numbers or equations where possible, but still try to be specific.

Hint: Does y approach a limit as $t \to \infty$? If so, what is this limiting value? How does this depend on r and K, if at all?

(b) Now suppose we introduce some sort of predation by a fixed number of predators. Then we could model this system with:

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y - Ey$$

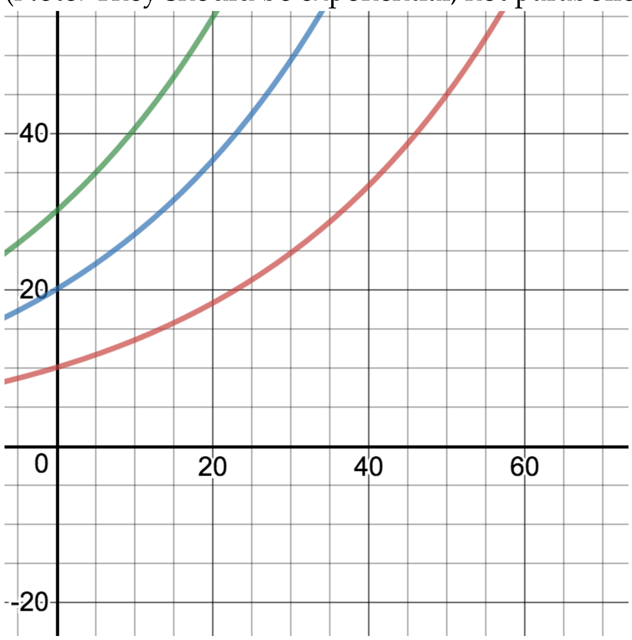
where E is another positive constant. What can you say about the qualitative behavior of $y(t)$ this time? As $t \to \infty$, does the population still approach a non-zero value or is it driven to extinction (i.e. $y \to 0$)? How does this depend on E, if at all?

Answer to Question 1. To check whether these are solutions, we plug the given functions $y(t)$ into both sides of the differential equation and check that they are equal.
For $y = 1 + t$, we get $\frac{dy}{dt} = \frac{d}{dt}[1 + t] = 1$ and

$$\frac{y^2 - 1}{t^2 + 2t} = \frac{(1+t)^2 - 1}{t^2 + 2t} = \frac{t^2 + 2t}{t^2 + 2t} = 1$$

since these two quantities are equal, $y(t) = 1 + t$ is a solution of the differential equation.
For $y = 1 + 2t$, we get $\frac{dy}{dt} = 2$, but $\frac{y^2-1}{t^2+2t} = \frac{4t^2+4t}{t^2+2t} \neq 2$, so this is not a solution.
For $y = 1$, we get $\frac{dy}{dt} = 0$ and $\frac{y^2-1}{t^2+2t} = 0$, so this is another solution.

Answer to Question 2.
Your graphs could look something like this:
(Note: They should be exponential, not parabolic)



(a) For the graph starting with $P = 10$, the slope increases as time increases. This makes sense, since more fish implies more reproduction, which implies a larger growth rate.
(b) For a set P value, the slope stays the same across the three graphs. This is because the slope of the graph is the rate of change $\frac{dP}{dt}$. Since the growth rate of the population depends only on the number of fish, not the time of day, it should be the same on all three graphs.

Answer to Question 3.
The rate of change $\frac{dP}{dt}$ should be a function of only the size of the population P, and not on the time t. For similar reasons to (2b), the rate of change is determined by the number of fish there are right now, not what time of day it is. This rules out all but the 2nd and 5th options. Note: When they first see this problem, many students are tempted to put $\frac{dP}{dt}$ = something involving t, because the rate is increasing in time. But since P is itself a function of t, this means $\frac{dP}{dt}$ depends *implicitly* on the value of t, so $\frac{dP}{dt}$ can still be increasing in time even without depending *explicitly* on it.
We can also rule out the 5th option using a sanity check of plugging in $P = 0$. Since $P = 0$ is the case where there are no fish, any reasonable model would also necessarily have $\frac{dP}{dt} = 0$, otherwise fish are being created out of nowhere. Since the fifth model would instead say $\frac{dP}{dt} = 1 > 0$, this model does not make sense.

$\frac{dP}{dt}$ should really depend linearly on P since increasing the number of fish by a constant factor would result in increasing reproduction by that exact same constant factor. I.e. if you have three as many fish parents you should have three times as many fish babies. This leaves $\frac{dP}{dt} = 2P$ as the only good model.

Answer to Question 4.
Here the terms that correspond to the interaction of the two species are those with *both* x and y. Whether the interactions are cooperative or competitive depends on the sign of the xy term.
Since (i) has positive coefficients in front of the xy terms, that means increasing the populations of both x and y have a positive effect on the rate at which both populations grow. This would correspond to the case of two cooperative species.
Similarly, since (ii) has negative coefficients in front of the xy terms, it means that increasing the numbers of x or y would have a negative effect on the rate of change of both species. This would correspond to the case of two competitive species.

Answer to Question 5.
(a) In looking for the qualitative behavior of solutions, it's often best to look at the sign of $\frac{dy}{dt}$ and how it changes. Here, we see that both the r and y terms are positive, so the sign of $\frac{dy}{dt}$ is determined by the sign of $1 - \frac{y}{K}$.
This means that is $y > K$, then $\frac{dy}{dt} < 0$, or in other words that any solution above the line $y = K$ will be decreasing.
If $y = K$, then $\frac{dy}{dt} = 0$, which means that the constant function $y(t) = K$ is in fact a solution. This is called an *equilibrium* solution of the differential equation.
If $0 < y < K$, then $\frac{dy}{dt} > 0$, so any solution below the line $y = K$ will be increasing.
Putting this all together, this means that any solution starting below $y = K$ will increase asymptotically up towards the limit $y = K$ as $t \to \infty$. Similarly, any solution starting above $y = K$ will decrease asymptotically towards K, while the solution starting at K will remain there. So no matter what the initial population $y(0)$ is, it will approach the limit of K when $t \to \infty$. In population growth models, K functions here as the carrying capacity. Note that the constant r doesn't affect the limiting value of y, just the rate at which solutions approach the limit K.
If this is confusing, don't worry, we'll go over more problems like this in Section 2.5

(b) This part works similarly to part a. Now, we can rearrange the terms as:

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K} - \frac{E}{r}\right)y$$

Now we see that the sign of $\frac{dy}{dt}$ depends purely upon the sign of $1 - \frac{y}{K} - \frac{E}{r}$.
First, if $E \geq r$, then $1 - E/r < 0$, so this term is always negative for any $y > 0$. This means that $\frac{dy}{dt}$ will always be negative, and so all solutions $y(t)$ will eventually decrease to the limit of 0 as $t \to \infty$, i.e. the population will go extinct.
However, if $E < r$ (i.e. if there is less predation), then we calculate that $\frac{dy}{dt} = 0$ when

$$1 - \frac{y}{K} - \frac{E}{r} = 0$$

which we can solve for y as:

$$y = K\left(1 - \frac{E}{r}\right)$$

This quantity is the new carrying capacity: Any population y starting at this equlbrium value will stay there for all times t since $\frac{dy}{dt} = 0$
We can further see that if $y > K(1 - E/r)$, then $\frac{dy}{dt} < 0$, so all solutions starting above this value will decrease asymptotically to this new carrying capacity. Similarly, all solutions starting below $y = K(1 - E/r)$, will increase asymptotically up to this new carrying capacity.