Math 2930 Quiz 4 Solutions

Week 12
November 9th, 2017

Question 1. Find the Fourier series of the following function on the interval $-\pi < x < \pi$

$$f(x) = |x|$$

Answer to Question 1.
Note: A lot of people tried to answer this question by integrating $f(x) = |x|$ as if it was just $f(x) = x$ which does not work!!
For example,

$$\int |x| dx \neq \frac{|x|^2}{2} + C$$

If you are going to integrate $|x|$ from $-\pi$ to π, you have to break it up as a piecewise function:

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

and integrate each part separately.

Since we are on the interval $-\pi < x < \pi$, this means that $L = \pi$.
It is also an extremely useful fact that $f(x) = |x|$ is *even*, since $f(-x) = f(x)$.

First, for the a_0 term, we calculate:

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x) dx = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| dx$$

we can split this up into two integrals and get:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{0} (-x) dx + \frac{1}{\pi}\int_{0}^{\pi} (x) dx = \frac{1}{\pi}\left[\frac{\pi^2}{2} + \frac{\pi^2}{2}\right] = \pi$$

Or, you could recognize that $f(x)$ is *even*, so this integral is:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| dx = \frac{2}{\pi}\int_{0}^{\pi} |x| dx$$

and then *since we are only working with positive values of* x, we can replace $|x|$ by x:

$$a_0 = \frac{2}{\pi}\int_{0}^{\pi} x dx = \frac{2}{\pi}\left(\frac{x^2}{2}\right)\Big|_0^{\pi} = \pi$$

Now for the a_n terms. We can calculate them from the formula:

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\cos(nx) dx$$

We could calculate this as:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} (-x)\cos(nx)\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\cos(nx)\,dx$$

but it is easier to notice that since $f(x)$ is even, both of these parts have the same value, so we can just calculate:

$$a_n = \frac{2}{\pi}\int_{0}^{\pi} |x|\cos(nx)\,dx = \frac{2}{\pi}\int_{0}^{\pi} x\cos(nx)\,dx$$

Using integration by parts,

$$\begin{aligned} U &= x & dV &= \cos(nx)\,dx \\ dU &= dx & V &= \frac{1}{n}\sin(nx) \end{aligned}$$

So this integral becomes:

$$\begin{aligned} a_n &= \left(\frac{2}{\pi}\right)\frac{x}{n}\sin(nx)\Big|_0^{\pi} - \frac{2}{n\pi}\int_0^{\pi}\sin(nx)\,dx \\ a_n &= 0 - \left(\frac{2}{n\pi}\right)\left(\frac{-1}{n}\cos(nx)\right)\Big|_0^{\pi} \\ a_n &= \frac{2(\cos(n\pi)-1)}{n^2\pi} \end{aligned}$$

which could also be written in the following two ways:

$$\begin{aligned} a_n &= \frac{2\left((-1)^n - 1\right)}{n^2\pi} \\ a_n &= \begin{cases} \frac{-4}{n^2\pi}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases} \end{aligned}$$

For the b_n terms, we could calculate them from the formula:

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\sin(nx)\,dx$$

But since $|x|$ is an *even* function, that means that $|x|\sin(nx)$ is an *odd* function, so integrating it from $-\pi$ to π will always cancel out to 0:

$$b_n = 0$$

Thus our Fourier series expansion is:

$$f(x) = |x| = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx) + b_n\sin(nx)$$

which could be written in any of the following equivalent ways:

$$\boxed{f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{-4}{(2n-1)^2\pi}\cos((2n-1)x)}$$

$$\boxed{f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{-2\left((-1)^n - 1\right)}{n^2\pi} \cos(nx)}$$

$$\boxed{f(x) = \frac{\pi}{2} + \sum_{n=1,3,5,7...}^{\infty} \frac{-4}{n^2\pi} \cos(nx)}$$

Note: Since our original function is *even*, you should know before even doing any integrals that your Fourier series should also only have *even* components, i.e. constants and cosines, with no sine terms.

This is a very useful "sanity check" on your answers to Fourier series problems: do they make sense in terms of representing even functions with only even functions and representing odd functions with only odd functions.