Math 2930 Discussion Notes
Resonance, BVPs

Week 9
October 19th, 2017

1 Resonance

Let's consider an undamped forced oscillator:

$$y'' + \omega_0^2 y = F_0 \cos(\omega t), \qquad y(0) = y'(0) = 0$$

The homogenous solution is:

$$y_h(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$$

1.1 $\boxed{\omega \neq \omega_0}$

In this case, we guess a particular solution of the form:

$$Y(t) = A\cos(\omega t) + B\sin(\omega t)$$

Plugging it into the original equation,

$$\begin{aligned} Y'' + \omega_0^2 Y &= F_0\cos(\omega t) \\ -\omega^2 A\cos(\omega t) + -\omega^2 B\sin(\omega t) + \omega_0^2 A\cos(\omega t) + \omega_0^2 B\sin(\omega t) &= F_0\cos(\omega t) \\ \left[A\omega_0^2 - A\omega^2\right]\cos(\omega t) + \left[B\omega_0^2 - B\omega^2\right]\sin(\omega t) &= F_0\cos(\omega t) \end{aligned}$$

Giving us the system of equations:

$$\begin{aligned} A(\omega_0^2 - \omega^2) &= F_0 \\ B(\omega_0^2 - \omega^2) &= 0 \end{aligned}$$

So

$$A = \frac{F_0}{\omega_0^2 - \omega^2}, \qquad B = 0$$

So our particular solution is

$$Y(t) = \frac{F_0}{\omega_0^2 - \omega^2}\cos(\omega t)$$

and the corresponding general solution is:

$$y(t) = y_h(t) + Y(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) + \frac{F_0}{\omega_0^2 - \omega^2}\cos(\omega t)$$

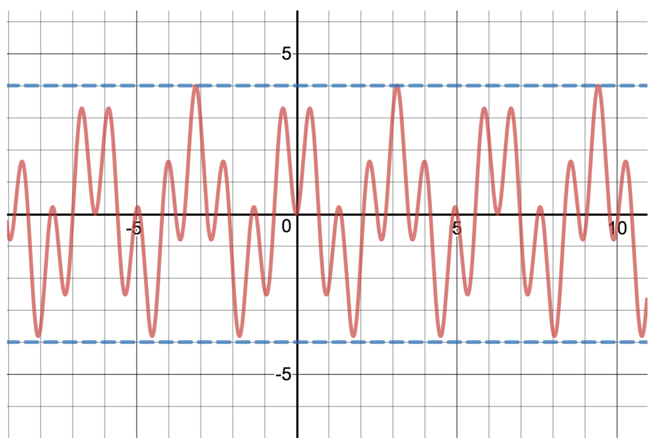
Plugging in the initial conditions to find c_1 and c_2,

$$\begin{aligned} y(0) &= c_1(1) + c_2(0) + \frac{F_0}{\omega_0^2 - \omega^2}(1) = 0 \\ c_1 &= -\frac{F_0}{\omega_0^2 - \omega^2} \\ y'(0) &= -\omega_0 c_1(0) + \omega_0 c_2(1) + \frac{F_0}{\omega_0^2 - \omega^2}(0) = 0 \\ c_2 &= 0 \end{aligned}$$

So the solution to this initial value problem is:

$$\boxed{\frac{F_0}{\omega_0^2 - \omega^2}\Big[\cos(\omega t) - \cos(\omega_0 t)\Big]}$$

An example graph of such a solution would be:



We see that this function has a sort of "amplitude" of $\frac{2F_0}{\omega_0^2 - \omega^2}$, represented by the dashed blue lines above.

We see that as $\omega \to \omega_0$, this "amplitude" increases to ∞. This essentially means that external forcing produces a stronger response when the forcing frequency ω is closer to the natural frequency ω_0.

What happens when $\omega = \omega_0$ exactly? This special case is what is known as *resonance*, and ω_0 is often referred to as the *resonant frequency* of the system.

1.2 $\boxed{\omega = \omega_0}$

If $\omega = \omega_0$, then we have to guess a particular solution of the form:

$$Y(t) = t\left[A\cos(\omega_0 t) + B\sin(\omega_0 t)\right]$$

where we have added an extra power of t to ensure that the particular solution is not contained in the homogenous solution.

We calculate its derivatives:

$$Y' = (A\cos(\omega_0 t) + B\sin(\omega_0 t)) + t(-A\omega_0 \sin(\omega_0 t) + B\omega_0 \cos(\omega_0 t))$$
$$Y'' = (-2A\omega_0 \sin(\omega_0 t) + 2B\omega_0 \cos(\omega_0 t)) + t(-A\omega_0^2 \cos(\omega_0 t) - B\omega_0^2 \sin(\omega_0 t)$$

Plugging it into the left hand side of the equation,

$$Y'' + \omega_0^2 Y = -2A\omega_0 \sin(\omega_0 t) + 2B\omega_0 \cos(\omega_0 t) = F_0 \cos(\omega_0 t)$$

Solving for A and B:

$$A = 0, \qquad B = \frac{F_0}{2\omega_0}$$

resulting in a particular solution of

$$Y = \frac{F_0}{2\omega_0} t \sin(\omega_0 t)$$

and a general solution of

$$y(t) = y_h(t) + Y(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{F_0}{2\omega_0} t \sin(\omega_0 t)$$

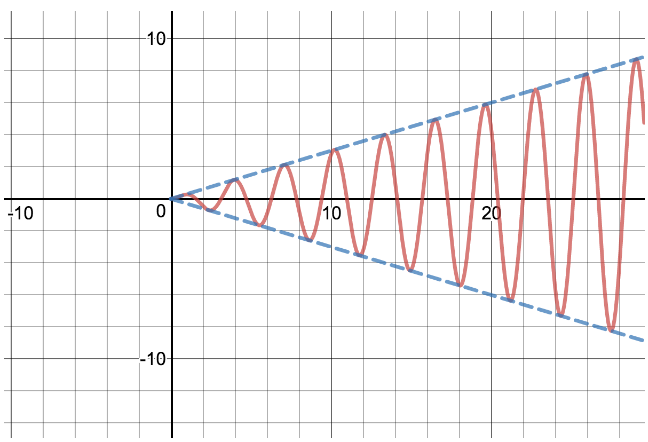
Plugging in the initial conditions,

$$y(0) = c_1 = 0$$
$$y'(0) = c_2 \omega_0 = 0$$

so the solution to the IVP is:

$$\boxed{y(t) = \frac{F_0}{2\omega_0} t \sin(\omega_0 t)}$$

An example graph of such a solution would be:



Where we see that y becomes unbounded as $t \to \infty$.

So the main point of these examples is that as the forcing frequency (ω) approaches the natural frequency (ω_0), the system exhibits a stronger response, eventually becoming unbounded at *resonance* when the forcing frequency ω is exactly the same as the natural frequency ω_0.

Note: While the examples here were undamped, similar ideas hold for the damped case.

2 Boundary Value Problems: A Linear Algebra Perspective

So by this point, we should be pretty used to initial value problems, which might look something like:

$$ay'' + by' + cy = 0, \qquad y(0) = d, \qquad y'(0) = e \tag{1}$$

The basic idea is that we find the general solution in the form:

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

Then we find c_1 and c_2 by plugging in the initial conditions, giving a system of 2 linear equations on 2 unknowns:

$$\begin{aligned} y(0) &= c_1 y_1(0) + c_2 y_2(0) = d \\ y'(0) &= c_1 y_1'(0) + c_2 y_2'(0) = e \end{aligned}$$

rewriting this system of equations in matrix form,

$$\begin{bmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} d \\ e \end{bmatrix}$$

If we compute the determinant of this matrix, we get the Wronskian evaluated at $x = 0$:

$$\det = y_1(0)y_2'(0) - y_1'(0)y_2(0) = W[y_1, y_2](0)$$

and it turns out that this Wronskian (*i.e.* the determinant) is never zero, so this system of equations always has a unique solution (c_1, c_2). However, for boundary value problems, this may not always be the case.

An example boundary value problem might look something like:

$$ay'' + by' + cy = 0, \qquad y(0) = d, \qquad y(L) = e \tag{2}$$

Note that Equation (2) is in *almost exactly* the same form as Equation (1). Except now the conditions are at two different locations: $x = 0$ and $x = L$, instead of both being at $x = 0$.

So we have the same general solution:

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

and we plug in the boundary conditions to get 2 linear equations on 2 unknowns:

$$\begin{aligned} y(0) &= c_1 y_1(0) + c_2 y_2(0) = d \\ y'(0) &= c_1 y_1(L) + c_2 y_2(L) = e \end{aligned}$$

rewriting this system of equations in matrix form,

$$\begin{bmatrix} y_1(0) & y_2(0) \\ y_1(L) & y_2(L) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} d \\ e \end{bmatrix}$$

Just like before. But this time, unlike with IVPs, it's actually possible for the determinant of this matrix to be 0, depending upon what y_1, y_2 and L are.

"Most" of the time, the matrix will still be invertible, and we'll have a unique solution (c_1, c_2) just like with IVPs.

But it is possible that the determinant is zero, in which case we will not have a unique solution (c_1, c_2). This then breaks down into two possible scenarios:

- There is no solution (c_1, c_2). *E.g.*

$$
\begin{aligned}
c_1 + c_2 &= 0 \\
c_1 + c_2 &= 1
\end{aligned}
$$

  where there is no possible way for c_1 and c_2 to satisfy both equations simultaneously.

- There are infinitely many solutions (c_1, c_2) *E.g.*

$$
\begin{aligned}
c_1 + c_2 &= 1 \\
2c_1 + 2c_2 &= 2
\end{aligned}
$$

  where these two equations are really the same equation, and so there are infinitely many choices of c_1 and c_2 that would lead to solutions of the BVP. When we learn about eigenvalue problems for BVPs, they can be thought of in a way as finding when this situation occurs.