Math 2930 Discussion Notes
Free Vibrations

Week 7
October 5th, 2017

1 Undamped Oscillations

Second-order constant coefficient equations show up in all kinds of physics/engineering applications. They come in the form:

$$my'' + \lambda y' + ky = 0$$

One of these equations is called *undamped* if $\lambda = 0$:

$$my'' + ky = 0$$

You may in fact recognize this as the equation of motion for a mass m attached to a spring with spring constant k. It's worth noting that in the vast majority of engineering applications, all of the coefficients in these equations are positive.

The characteristic polynomial for this equation is:

$$mr^2 + k = 0$$

solving for r,

$$r^2 = \frac{-k}{m}$$

and whenever both m and k are positive, these roots are purely imaginary:

$$r = \pm\sqrt{\frac{k}{m}}$$

which results in a general solution of:

$$y(t) = C_1 \cos\left(\sqrt{\frac{k}{m}}t\right) + C_2 \sin\left(\sqrt{\frac{k}{m}}t\right)$$

So regardless of the initial conditions, we get a steady oscillation with period $2\pi\sqrt{\frac{m}{k}}$.

2 Damped Oscillations

Now, in many engineering applications, steady oscillations are just an idealized scenario, and many practical scenarios have a non-negligible friction or drag force. These result in *damping* $\lambda y'$ term in the equation, where λ is another positive constant:

$$my'' + \lambda y' + ky = 0$$

Solving for the roots of the characteristic polynomial with the quadratic formula,

$$mr^2 + \lambda r + k = 0$$

$$r = \frac{-\lambda \pm \sqrt{\lambda^2 - 4km}}{2m}$$

There are three different scenarios that can occur based upon the sign of the term inside of the square root.

2.1 Overdamped

If $\lambda^2 > 4km$, then the term inside the square root is positive, and so both roots are real. In fact, we see that:

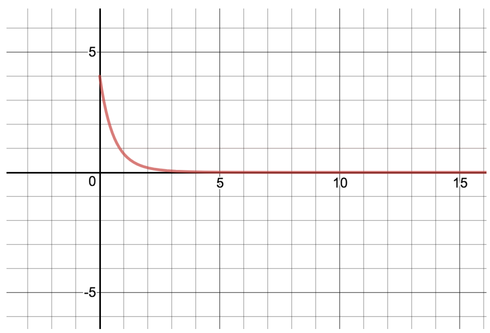
$$\sqrt{\lambda^2 - 4km} < \sqrt{\lambda^2} = |\lambda|$$

which means that both of the roots are actually negative. In this case, the general solution will be

$$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

where both r_1 and r_2 are negative, so y just decreases exponentially regardless of the initial conditions. In this case, the system is called *overdamped*.

A graph of such a solution will generically look something like:



2.2 Critically damped

If $\lambda^2 = 4km$, then the term inside the square root is exactly zero, and we have a repeated root of:

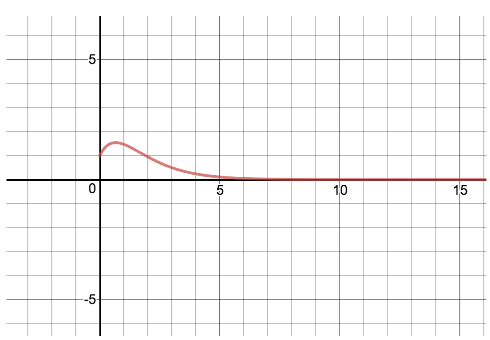
$$r = \frac{-\lambda}{2m}$$

which is always negative. The corresponding general solution is:

$$y(t) = C_1 e^{-\lambda t/2m} + C_2 t e^{-\lambda t/2m}$$

As $t \to \infty$, you will still get that $y \to 0$, but unlike the overdamped case, this will not always happen monotonically. For certain initial conditions, the solutions will got some sort of local minimum/maximum, and *then* decrease exponentially. This case

A graph of such a solution might look something like:



2.3 Underdamped

The final case occurs when $\lambda^2 < 4km$, in such a case, we get complex roots with real part $\frac{-\lambda}{2m}$. The corresponding general solutions are of the form:

$$y(t) = C_1 e^{-\lambda t/2m} \cos(\omega t) + C_2 e^{-\lambda t/2m} \sin(\omega t)$$

These solutions generically look like decaying oscillations. A graph of one might look something like: