Math 2930 Discussion Notes
2nd-order homogenous

Week 5
September 21st, 2017

1 Constant Coefficients

First, I wanted to remind everyone of how to solve second-order linear homogenous equations with *constant* coefficients. Such equations look like:

$$ay'' + by' + cy = 0 \tag{1}$$

The basic idea is to use the *ansatz* that the solution is something of the form $y(t) = e^{rt}$. This yields the *characteristic polynomial*:

$$ar^2 + br + c = 0 \tag{2}$$

There are then three possibe cases depending upon the roots of (2), leading to different forms of the general solution:

Distinct real roots	Complex roots	Repeated real roots
$r_1, \quad r_2$	$a \pm bi$	$r_1, \quad r_1$
$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$	$y(t) = C_1 e^{at}\cos(bt) + C_2 e^{at}\sin(bt)$	$y(t) = C_1 e^{r_1 t} + C_2 t e^{r_1 t}$

However, this approach only works when the coefficients in (1) are *constant*, and not functions of t themselves. If the coefficients are not constant, solving the equation is much trickier, so I want to go over an example of that in greater detail.

2 Example: Turbulent Flow

One way of solving second-order ODEs with variable coefficients is the method of *reduction of order*. This is a topic that often confuses students, so I wanted to take the time to walk through an example of how it can be used in detail. This specific problem was actually one of the harder problems from a prelim in this course in a past semester.

The turbulent flow of a uniform stream of a liquid past a circular cylinder is described by the differential equation:

$$y'' + a(xy' + y) = 0 \tag{3}$$

where a is a parameter depending on the nature of the liquid.

3 Initial Solution

The first step is to show that

$$y(x) = e^{-ax^2/2}$$

is a solution of Equation (3).

To do that, we will plug this given function $y(x)$ into Equation (3) and check that both sides are equal. We calculate y' and y'' as:

$$y'(x) = -axe^{-ax^2/2}$$
$$y''(x) = -ae^{-ax^2/2} + (ax)^2 e^{-ax^2/2}$$

Then plugging this into Equation (3),

$$y'' + a(xy' + y) = \left(-ae^{-ax^2/2} + (ax)^2 e^{-ax^2/2}\right) + a\left(x\left[-axe^{ax^2/2}\right] + e^{ax^2/2}\right)$$

Which simplifies to

$$\left(-a + (ax)^2 + a\big(x[-ax] + 1\big)\right)e^{-ax^2/2} = \left(-a + (ax)^2 - (ax)^2 + a\right)e^{-ax^2/2} = 0$$

So $y(x) = e^{-ax^2/2}$ is in fact a solution of Equation (3).

4 Reduction of Order

OK, so we have one solution $e^{-ax^2/2}$. But Equation (3) is a linear (homogenous) second-order differential equation, and we expect linear second-order equations to have general solutions of the form:

$$y(x) = C_1 y_1(x) + C_2 y_2(x)$$

Since Equation (3) is linear, we can use $y_1(x) = e^{-ax^2/2}$, but since we do not have constant coefficients, we have no idea what $y_2(x)$ might look like.

The idea of the method of *reduction of order* is to instead look for a general solution of the form:

$$y(x) = y_1(x)v(x) \left(= e^{-ax^2/2} v(x)\right)$$

Plugging this into our original Equation (3), we will be able to get a new differential equation for $v(x)$. So we calculate:

$$y'(x) = \big[y_1(x)v(x)\big]' = y_1'(x)v(x) + y_1(x)v'(x)$$
$$y''(x) = \big[y_1(x)v(x)\big]'' = y_1''(x)v(x) + 2y_1'(x)v'(x) + y_1(x)v''(x)$$

Plugging these into Equation (3),

$$y'' + a(xy' + y) = \big(y_1''v + 2y_1'v' + y_1v''\big) + a\big(xy_1'v + xy_1v' + y_1v\big) = 0$$

Let's rearrange this according to the v term,

$$(y_1)v'' + (2y_1' + axy_1)v' + (y_1'' + axy_1' + ay_1)v = 0$$

If we look carefully at the coefficient of the v term, we see that it is actually our original differential equation (3). Because we already determined that $y_1(x) = e^{-ax^2/2}$ is a *solution* of this equation, we have that this whole term equals zero and thus cancels out, leaving:

$$y_1v'' + (2y_1' + axy_1)v' = 0 \tag{4}$$

And now that the v term has cancelled out, leaving only its derivatives, we can use the substitution $w(x) = v'(x)$, giving:

$$y_1 w' + \left(2y_1' + axy_1\right)w = 0 \tag{5}$$

which is a first-order linear equation for $w(x)$, which we can solve using integrating factors. Plugging in that $y_1(x) = e^{-ax^2/2}$,

$$\left(w'(x) + axw(x)\right)e^{-ax^2/2} = 0$$

Since $e^{-ax^2/2} \neq 0$, we can divide it out of the equation, yielding

$$w'(x) + axw(x) = 0 \tag{6}$$

For Equation (6), the integrating factor is $\mu(x) = e^{-ax^2/2}$,

$$e^{-ax^2/2}w'(x) + axe^{-ax^2/2}w(x) = \left(e^{-ax^2/2}w(x)\right)' = 0$$

Integrating both sides,

$$e^{-ax^2/2}w(x) = C$$

So converting back to $v(x)$,

$$w(x) = v'(x) = Ce^{ax^2/2}$$

Integrating both sides again,

$$v(x) = C_2 \int e^{ax^2/2} dx + C_1$$

So we find that the general solution is:

$$y(x) = e^{-ax^2/2}v(x) = C_1 e^{-ax^2/2} + C_2 e^{-ax^2/2}\left[\int e^{ax^2/2} dx\right]$$

So reduction of order managed to reduce our original equation (3), which was a 2nd-order linear equation, into (6), which was a 1st-order linear equation that we know how to solve. This turns out to work in general, but it does require already knowing a solution to the original equation, which you may not always have.