Math 2930 Discussion Notes
Euler's method

Week 4
September 14, 2017

1 Euler's Method Basics

The main idea of Euler's method is that we are given a first-order ordinary differential equation of the form:

$$\frac{dy}{dt} = f(t, y), \qquad y(t_0) = y_0$$

and we want to calculate an *approximate* solution of this equation.

With Euler's method, we start at the point (t_0, y_0) and want to approximate the value of $y(t)$ at some time t_1. The approach of Euler's method is to approximate the solution as a straight line with slope given by $f(t_0, y_0)$. Then we figure out our approximate value y_1 by calculating the value of this line at t_1. Algebraically, this process would be:

$$y_1 = y_0 + (t_1 - t_0)f(t_0, y_0)$$

We can then repeat this process for more steps, using (y_1, t_1) to calculate (y_2, t_2), then using that to calculate (y_3, t_3) and so on. So more generally we have:

$$y_{n+1} = y_n + (t_{n+1} - t_n)f(t_n, y_n)$$

Often the timestep $t_{n+1} - t_n$ is uniform, and so we call it h, in which case we have:

$$y_{n+1} = y_n + hf(t_n, y_n)$$

2 Euler's Method Example

Today I want to go over a basic example with Euler's method that I think is particularly instructive.

Let's consider the example of:

$$\frac{dy}{dt} = -1000y, \qquad y(0) = 1$$

By now, you'll probably recognize that the exact solution is $y(t) = e^{-1000t}$. This function starts out at 1, and very quickly approaches 0 as t increases. We might expect that Euler's method will then give us a sequence of straight lines that resemble this curve, but let's see what actually happens.

Let's use a "small" stepsize of $h = 0.01$, then our Euler's method formula will become:

$$y_{n+1} = y_n + 0.01f(t_n, y_n) = y_n + 0.01(-1000y_n) = y_n - 10y_n = -9y_n$$

We start with $y_0 = y(0) = 1$. We calculate the first few steps:

$$y(0.01) \approx y_1 = -9y_0 = (-9)(1) = -9$$
$$y(0.02) \approx y_2 = -9y_1 = (-9)(-9) = 81$$
$$y(0.03) \approx y_3 = -9y_2 = (-9)(81) = -729$$

And so on. The exact solution is supposed to go to 0 very quickly, but this approximation keeps switching sign and is *increasing* exponentially instead of decreasing exponentially. Why is this going on? Why is Euler's method not "working" here?

The answer is that we are not using a small enough timestep h. I picked this equation out because it is what is known as a *stiff* equation, basically meaning that it requires a very small timestep h to be computed accurately. It turns out that if we use an even smaller value of h then the approximate solution will behave correctly.

The moral of the story here is that Euler's method "works" in the sense that it converges to the correct solution as $h \to 0$. But that doesn't mean it is necessarily accurate for a given h. The ideal way to use numerical methods such as Euler's method is using what's often called "grid refinement": using a series of decreasing values of h and checking that the solution is converging as $h \to 0$.