Math 2930 Discussion Notes
Equilibria and Stability

Week 3
September 7, 2017

Example: Logistic Equation with Predation

Today, we're going to focus on *autonomous* first-order ODEs, which are equations of the form:

$$\frac{dy}{dt} = f(y)$$

That is, equations where $\frac{dy}{dt}$ depends only on y, and not (explicitly) on t. Let's work with an example from Week 1's worksheet.

Let $y(t)$ be the population of some species in an environment with limited resources. Suppose this species is subject to predation by a fixed number of predators. Then a differential system that could be used to describe this situation is:

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y - Ey \tag{1}$$

where the parameters r, K, and E are all positive constants. Equation (1) is known as the *Schaefer* model after the biologist M.B. Schaefer.

Here, the $-Ey$ term represents predation, with a higher value of the parameter E corresponding to a higher level of predation, and the special case $E = 0$ corresponding to no predation. The other term is a standard logistic growth model with carrying capacity K.

Note that this is in fact a separable equation, which we know how to solve explicitly by separating variables and integrating:

$$\int \frac{dy}{r\left(1 - \frac{y}{K}\right)y - Ey} = \int dt$$

However, this integral on the left hand side would be a pain to compute. It would require using a partial fraction decomposition, and the number of constants involved would lead to many opportunities for small mistakes in the integration. Even if all the integration was done correctly, this would leave us with an *implicit* form of the solution. If we wanted to solve for y as an *explicit* function of t, that would require additional algebra, and may in fact be impossible.

But we'd still like to understand something about the general behavior of solutions without going through the hassle of solving this equation explicitly. One useful thing about autonomous equations is that for any value of y_0 where $f(y_0) = 0$, then the constant solution $y(t) = y_0$ is in fact a solution to the equation. This is because for constant functions, $\frac{dy}{dt} = 0$, so both sides of the equation match. These constant solutions are called *equilibria*, and can generally be found by setting $\frac{dy}{dt} = 0$ and solving algebraically.

Note that this process only works for *autonomous* ODEs. On the first homework, with problem 1.1.22, many students tried setting $\frac{dy}{dt} = 0$ and solved for $y = t - 2$, and said that all solutions approached this function asymptotically. But the solutions actually approach $y = t - 3$ asymptotically, which you can find by solving the equation explicitly.

With that disclaimer out of the way, let's find the equilibrium solutions for the Schaefer model. We set

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y - Ey = 0$$

Rearranging terms,

$$\left(r - \frac{ry}{K} - E\right)y = 0$$

We see that this has two solutions:

$$r - \frac{ry}{K} - E = 0 \quad \text{and} \quad y = 0$$

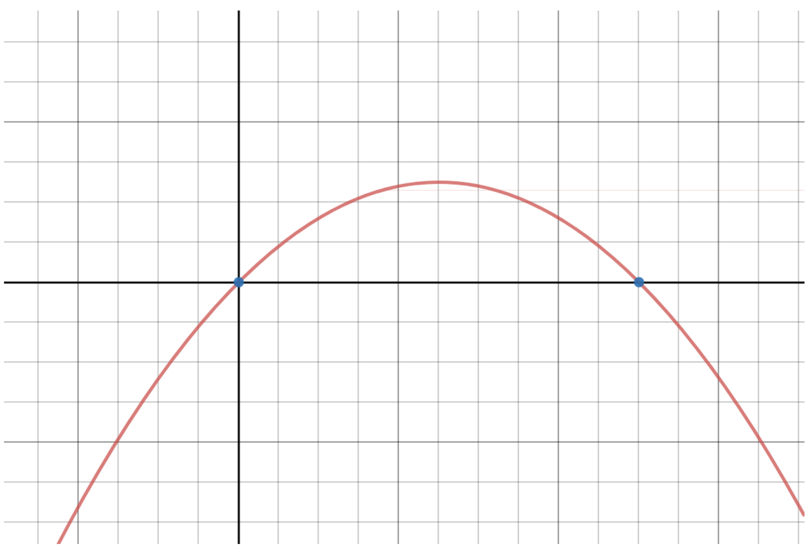
And the first case can be solved to get:

$$y = K\left(1 - \frac{E}{r}\right)$$

So these are the two equilibria of the Schaefer model.

OK, so those are two constant solutions of the equation, and if our initial condition is exactly one of those two constants, y will stay at that constant value for all time. But what happens if $y(0)$ doesn't fall on one of these values? We can figure out this by looking at the sign of $\frac{dy}{dt}$ to see where y is increasing or decreasing. It turns out that there are two main cases here: when $E < r$, and $E \geq r$. We'll need to handle these two cases differently., so let's start with the case when $E < r$.

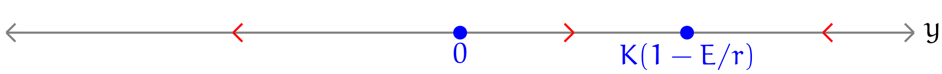
To help us see how $\frac{dy}{dt}$ behaves, we'll graph $f(y)$ versus y. We see in this case that $f(y)$ is a parabola, with the quadratic term having a negative coefficient, so the parabola should be curving downward. We also know where the two zeroes of this parabola are, since these are exactly the equilibria we already found. So a sketch of $f(y)$ would look like:



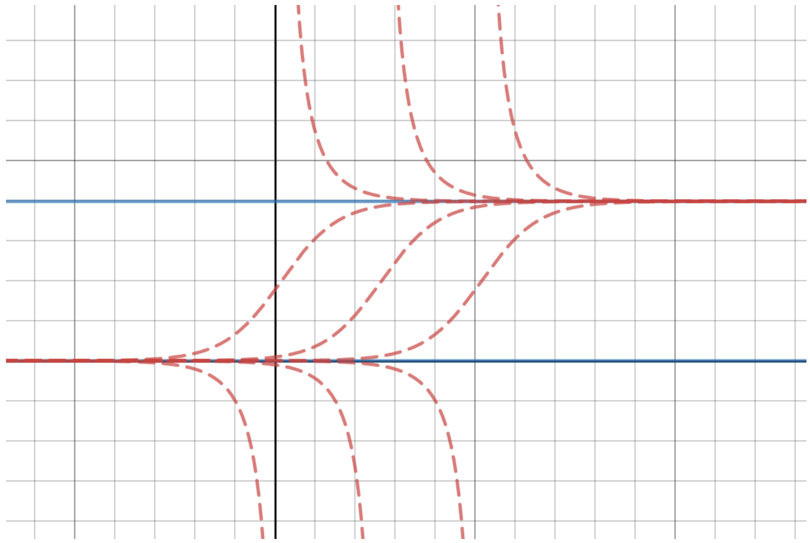
Here, $f(y)$ is the vertical axis, y is the horizontal axis, and the blue points are the two equilibria.

This graph tells us where $\frac{dy}{dt}$ is positive, negative, or zero. This in turn tells us where $y(t)$ is increasing and decreasing.

We can more succintly summarize the information in this graph into what is called a phase line:



A sketch of some solutions on the $y(t)$ plane would thus look like:

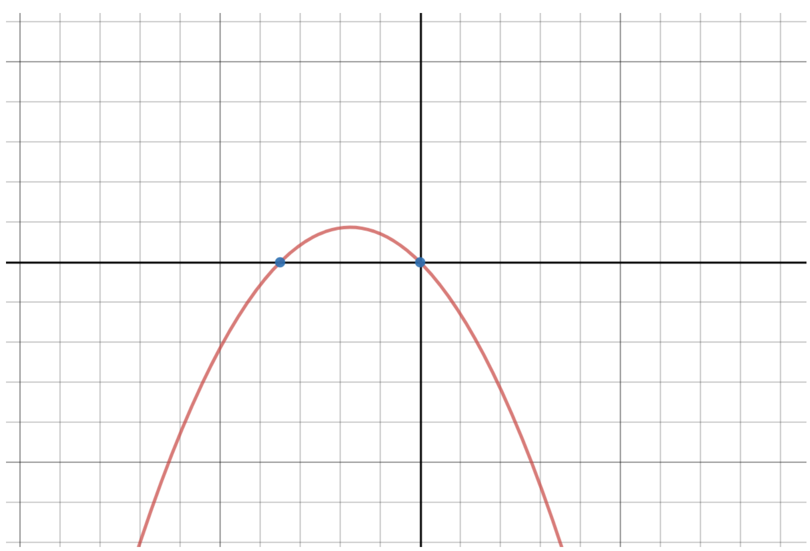


The vertical axis is now y (be careful: y was the horizontal axis in the previous graph, but the vertical axis in this one). The horizontal axis is t. The solid blue lines are the equilibrium solutions, and the dashed red lines are the non-equilibrium solutions.

So how can we interpret this graph? For everything below the t-axis, this would correspond to a negative population, which isn't meaningful. But we see that for all positive initial conditions, the population eventually approaches the equilibrium solution $y = K(1 - E/r)$. This means that for "small" predation ($E < r$), then this behaves like a new carrying capacity. Initial populations that are larger than the equilibrium decrease until they approach the equilibrium, and vice versa for (positive) initial conditions below the equilibrium.

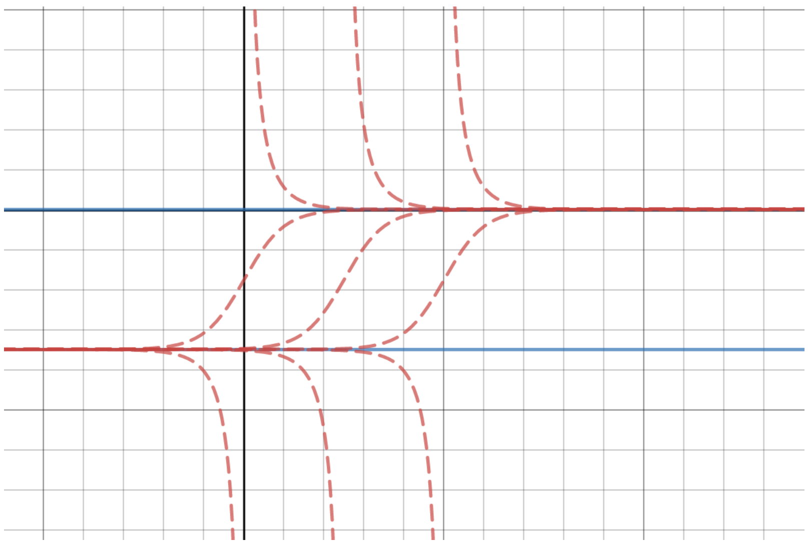
Now things are slightly different for the case when there is more predation and $E \geq r$. In this case, we still have the same expressions for the equilibria, but now the nonzero equilibrium is in fact negative rather than positive, which will change the overall behavior of the system. (To be precise, if $E = r$ there is only the equilibrium at zero.)

In this case, the graph of $f(y)$ would look something like:



Here, $f(y)$ is the vertical axis, y is the horizontal axis, and the blue points are the two equilibria.

In this case, we see that the negative equilibria is unstable, and zero is now stable. A sketch of some solutions on the $y(t)$ plane would thus look like:



The vertical axis is y, and the horizontal axis is t. The solid blue lines are the equilibrium solutions, and the dashed red lines are the non-equilibrium solutions.

In this case, zero is a stable equilibrium, and everything below it is nonphysical. Since all non-negative initial conditions approach $y = 0$ as $t \to \infty$, this can be interpreted as the Schaefer model saying that the population will go extinct in the case of $E \geq r$.