Math 2930 Discussion Notes
Wave Equation:
d'Alembert's Formula

Week 13
November 16th, 2017

1 d'Alembert's Formula

The wave equation is a second-order linear PDE:

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

usually with some sort of initial conditions, *e.g.*:

$$\begin{aligned} u(x,0) &= f(x) \\ u_t(x,0) &= 0 \end{aligned}$$

As with the heat equation, we can solve this using separation of variables, looking for solutions of the form $u(x,t) = X(x)T(t)$.

While this works, there is also another way of solving the wave equation. It turns out that solutions $u(x,t)$ to the wave equation can always be written in the form

$$u(x,t) = F(x+at) + G(x-at)$$

for some functions F and G. The exact reasons this works will be part of this week's homework assignment, but you can think of this as writing your solution in terms of a wave F that moves to the left and a wave G that moves to the right.

We can check that this in fact solves the wave equation by just plugging it in:

$$\begin{aligned} u &= F(x+at) + G(x-at), \\ u_t &= aF'(x+at) - aG'(x-at), \\ u_{tt} &= a^2F''(x+at) + a^2G''(x-at), \end{aligned} \qquad \begin{aligned} u &= F(x+at) + G(x-at) \\ u_x &= F'(x+at) + G'(x-at) \\ u_{xx} &= F''(x+at) + G''(x-at) \end{aligned}$$

$$u_{tt} = a^2 u_{xx}$$

So now that we have a solution, we can plug it into the initial conditions to find F and G. For the initial position:

$$\begin{aligned} u(x,t) &= F(x+at) + G(x-at) \\ u(x,0) &= F(x) + G(x) = f(x) \end{aligned}$$

For the initial velocity:

$$\begin{aligned} u_t(x,t) &= aF'(x+at) - aG'(x-at) \\ u_t(x,0) &= aF'(x) - aG'(x) = 0 \end{aligned}$$

So we have two equations for our two unknown functions F and G:

$$\begin{aligned} F(x) + G(x) &= f(x) \\ aF'(x) - aG'(x) &= 0 \end{aligned}$$

The second equation then says that

$$F'(x) = G'(x)$$

so if the derivative of F and G are the same, then F and G are the same up to a constant C:

$$F(x) = G(x) + C$$

Plugging this into the first equation,

$$2G(x) + C = f(x)$$

We can then solve for F and G as follows:

$$\begin{aligned} G(x) &= \frac{f(x) - C}{2} \\ F(x) = G(x) + C &= \frac{f(x) + C}{2} \end{aligned}$$

So then our solution $u(x, t)$ is:

$$\begin{aligned} u(x, t) = F(x + at) + G(x - at) &= \frac{f(x + at) + C}{2} + \frac{f(x - at) - C}{2} \\ &= \frac{1}{2}\left(f(x + at) + f(x - at)\right) \end{aligned}$$