Math 2930 Discussion Notes
Even/Odd Fourier Series

Week 11
November 2nd, 2017

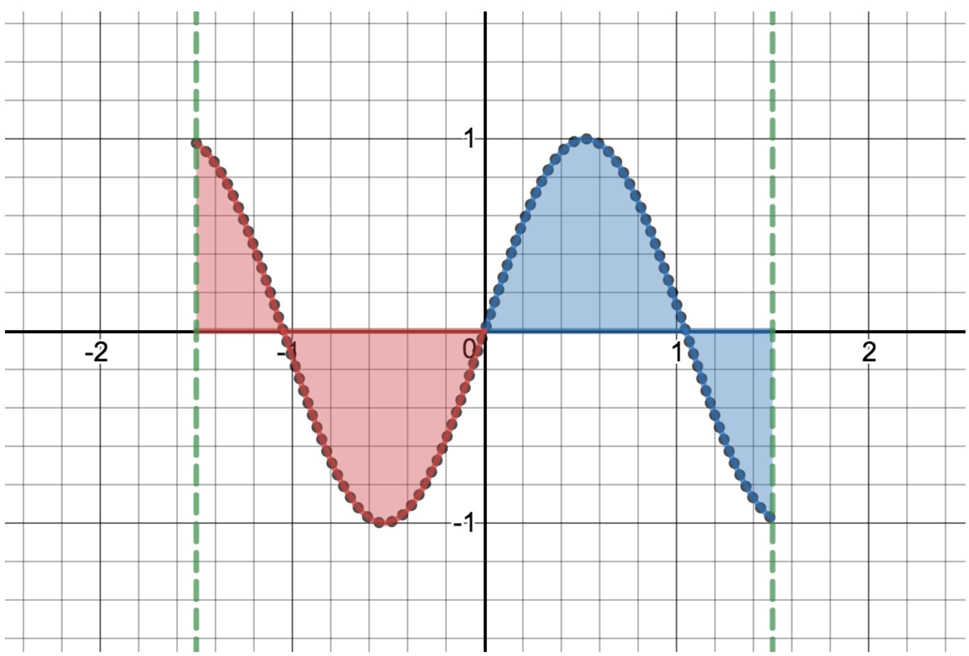
1 Integrals of Even/Odd Functions

I want to take a moment to remind everyone of some Calculus I tricks that some people may have forgotten, but are very relevant to the Fourier series problems we're doing in the course right now.

If you take the integral of *any* odd function f, and integrate it over *any* symmetric interval $[-L, L]$, then it works out that this integral is zero:

$$f(x) \text{ odd} \implies \int_{-L}^{L} f(x)dx = 0$$

While this can be formally derived by splitting up the integral into two pieces and using a substitution, I personally find it easier to remember this fact in terms of a picture like this one:

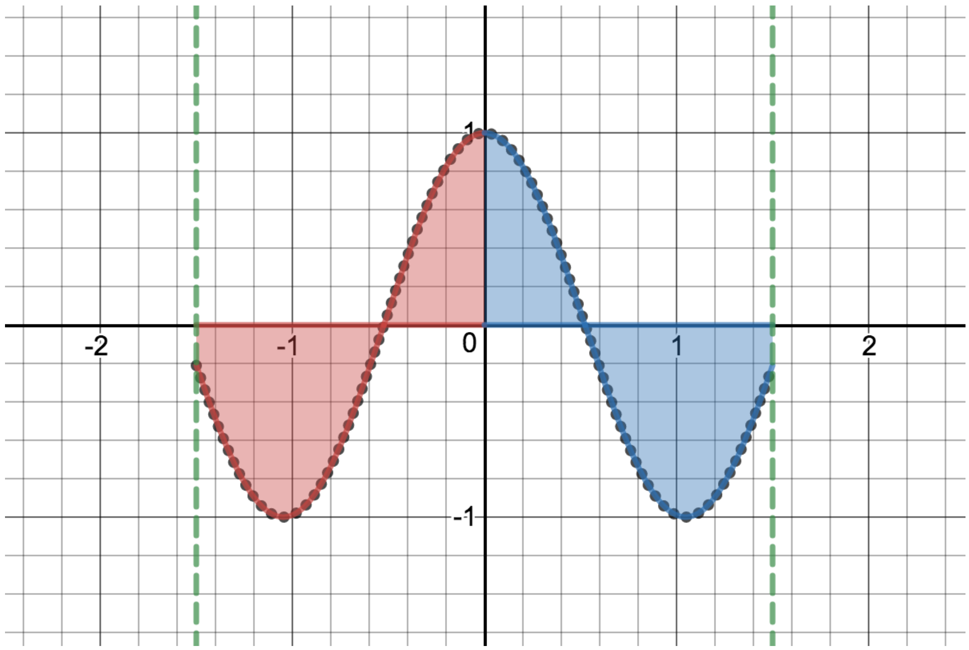


The point here is that the red region and blue region have exactly the same area, but are flipped versions of each other because f is an odd function. So when integrating f, you are finding the signed area under the curve, and these two will cancel each other out.

There's an analogous trick for integrating even functions as well. In this case, integrating an even function over $[-L, L]$ is the same as integrating the function over $[0, L]$ and then multiplying by a factor of 2:

$$f(x) \text{ even} \implies \int_{-L}^{L} f(x)dx = 2\int_{0}^{L} f(x)dx$$

In this case, the corresponding picture would be:



Here we can see that the red and blue regions have the exact same area, so to calculate the integral we can just find the area of one of them and then double it.

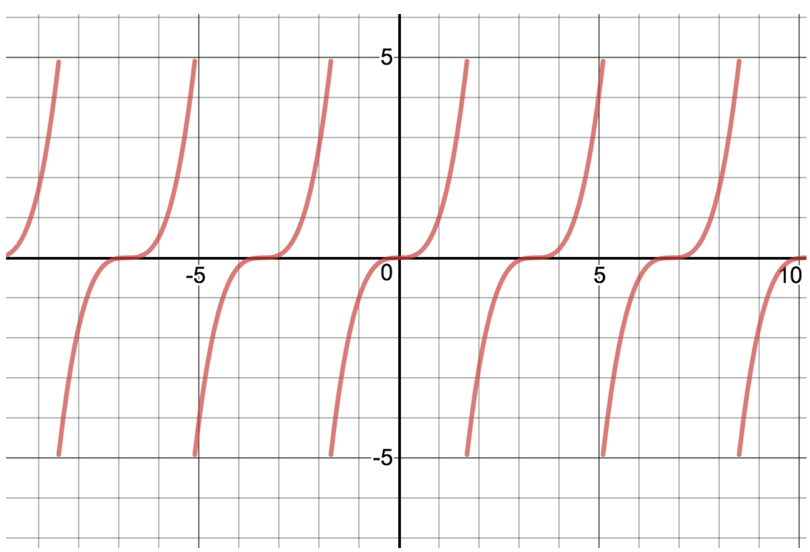
2 Even/Odd Fourier Series

These tricks turn out to be very useful in computing the coefficients for Fourier series expansions, because sine and cosine are odd and even functions, respectively.

Let's start with an example, say:

$$f(x) = x^3, \qquad x \in [-L, L]$$
$$f(x + 2L) = f(x)$$

Its graph would look something like:



Then we would calculate its Fourier series coefficients with the formulas:

$$a_n = \frac{1}{L}\int_{-L}^{L} x^3 \cos\left(\frac{n\pi x}{L}\right) dx$$
$$b_n = \frac{1}{L}\int_{-L}^{L} x^3 \sin\left(\frac{n\pi x}{L}\right) dx$$

Now, for the first of these integrals, we note that x^3 is an odd function, and cos is an even function, so this means that:

$$a_n = \frac{1}{L}\int_{-L}^{L} (\text{odd})(\text{even}) dx = \frac{1}{L}\int_{-L}^{L} (\text{odd}) dx = 0$$

If you're having trouble understanding why an odd function times an even function is an odd function, you can remember it this way:

- Odd functions behave like odd powers of x, i.e. x, x^3, x^5, ...
- Even functions behave like even powers of x, i.e. x^0, x^2, x^4, ...
- And to *multiply* powers of x, then you *add* the exponents.
- So (even function)(odd function) = (odd function)

But anyway, this all means that we can make calculating our Fourier series coefficients much simpler by taking advantage of whether $f(x)$ is odd or even. So for our example of $f(x) = x^3$, we get:

$$a_n = \frac{1}{L}\int_{-L}^{L} x^3 \cos\left(\frac{n\pi x}{L}\right) dx = 0$$
$$b_n = \frac{1}{L}\int_{-L}^{L} x^3 \sin\left(\frac{n\pi x}{L}\right) dx = \frac{2}{L}\int_{0}^{L} x^3 \sin\left(\frac{n\pi x}{L}\right) dx$$

And here's another way of thinking of this idea:

If $f(x)$ is an odd function, its Fourier series should only have odd terms in it, and thus should be a sum of only sine functions:

$$\text{odd } f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

And similarly, an even function should only have even terms in its Fourier series, meaning it should be a sum of cosine functions (maybe with a constant in front), i.e.:

$$\text{even } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$