Math 2930 Discussion Notes

Introduction to Differential Equations

Week 1

August 24, 2017

1 Introduction

The vast majority of this class will be focused on techniques for finding analytic solutions to differential equations. So today I want to focus on how to reason qualitatively about differential equations.

2 What are solutions?

Everyone should be familiar with basic *algebraic* equations like:

$$x^2 - 3x + 2 = 0 \tag{1}$$

And how to solve them, e.g. by factoring or the quadratic formula. By solution here we mean a number x where when we plug it into both sides of the equation, the two sides match.

Now let's compare with an example of a *differential* equation:

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = 0 \tag{2}$$

(It actually turns out this is closely related to the algebraic equation above, we'll learn more about this later). In this case, a *solution* y is no longer a number like before, but in fact a function of the independent variable t. I want to draw attention to the fact that we will often write just y as in the differential equation above, and not $y(t)$ to save space. It's important to remember that even though we're only writing y we really mean y is a function of t, and not just a variable with some constant value.

Fortunately, many of the same ideas about algebraic equations carry over. Like algebraic equations, differential equations may have exactly one solution, many solutions, or no solutions at all. And even if we don't know how to find a solution, we can still check if a given function is a solution or not by plugging it in and seeing if both sides are equal. We could in principle figure out the solutions to differential equations by guessing and checking with a bunch of different functions, but as with algebraic equations, this is extremely inefficient as a way of finding solutions unless we get very, very lucky with our guesses.

But, it is still a useful way of checking ourselves if we have a function that we have reason to believe is a solution. For instance, if I gave you the function $y(t) = \cos(t)$ and asked you to check if it was a solution to Equation (2), you would plug it into both sides:

$$\begin{aligned} \frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y &= 0 \\ \frac{d^2}{dt^2}\big[\cos(t)\big] - 3\frac{d}{dt}\big[\cos(t)\big] + 2\big[\cos(t)\big] &= 0 \\ -\cos(t) - 3(-\sin(t)) + 2(\cos(t)) &= 0 \\ \cos(t) + 3\sin(t) &= 0 \end{aligned}$$

Note that the two sides are equal for *some* values of t. But they definitely are not equal for *all* values of t. In other words, they are not the same functions. Therefore this would not be a solution to Equation (2).

On the other hand, if I gave you the function $y(t) = e^{2t}$ and asked you to check if that was a solution to Equation (2), then you would get:

$$\begin{aligned} \frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y &= 0 \\ \frac{d^2}{dt^2}\big[e^{2t}\big] - 3\frac{d}{dt}\big[e^{2t}\big] + 2\big[e^{2t}\big] &= 0 \\ 4e^{2t} - 3(2e^{2t}) + 2(e^{2t}) &= 0 \\ (4 - 6 + 2)e^{2t} &= 0 \\ 0 &= 0 \end{aligned}$$

Since these two sides are always equal, this means that e^{2t} is in fact a solution to Equation (2).

In the rest of the course, you generally won't be required to plug solutions directly into differential equations all that often, but there are a few reasons I wanted to go over this:

(1) So that you hopefully keep in mind what being a solution of a differential equation really means

(2) This is a useful way of checking your answers. This is a really good idea to keep in mind in case you have extra time left on quizzes or exams. Also, I've seen that some of the textbook's answers to HW questions are wrong. So if you keep getting an answer that disagrees with the textbook, this would be a good way of determining whether you or the textbook is correct.

3 Qualitative Reasoning

OK, so we know what solutions to differential equations are, and how to check if we have a solution, but we'd like to hopefully develop some sort of intuition for what differential equations *mean*, and why they're useful.

Let's start with an example straight out of high-school and first-year college physics: a mass m attached to a spring with spring constant k. Let's label it's displacement from equilibrium as $x(t)$, which means its velocity is $x'(t)$.

You've probably seen this system many, many times in physics, and hopefully know that the resulting motion is sinusoidal. So to make things interesting, let's say that instead of the spring exerting a force of kx according to Hooke's law, it exerts a force of kx^3 instead. In this case, we don't know exactly what the resulting motion would be (it wouldn't be the same sin and cos functions as before), but we can still determine some things about its motion.

From physics, we know that energy in the system is conserved. The kinetic energy is given by $K = \frac{1}{2}mv^2 = \frac{1}{2}m\left(x'\right)^2$. And the potential energy will be given by $U = \frac{1}{4}kx^4$.

So conservation of energy tells us that for some constant C,

$$\frac{1}{2}m(x')^2 + \frac{1}{4}kx^4 = C \tag{3}$$

This is then a differential equation that the displacement $x(t)$ of the spring must be a solution of. Since x is a function of only one independent variable t, this is an *ordinary* differential equation, usually abbreviated as an *ODE*. Since the highest order derivative that shows up in the equation is the first derivative, this is a *first-order* ODE. And because the x and x' terms are taken to powers other than one, this is a *nonlinear* equation. C is a positive constant that we could figure out from the initial position and velocity, but let's keep things simple and leave it alone for now. The important part is that it is constant, i.e. it does not change with time.

Right now, we haven't yet learned how to actually solve a differential equation like this. But we can still figure out some basic facts about solutions even if we don't know exactly what they are. So let's take another look at (3):

$$\frac{1}{2}m(x')^2 + \frac{1}{4}kx^4 = C$$

Even though x and x' are quantities that vary in time, this differential equation is about relating x and x' to each other, rather than relating both of them to t, which is the usual idea in first-year calculus. This is a very common approach with differential equations, that we look to relate the evolution of the system (in terms of $\frac{dx}{dt}$ or other derivatives), to the state of the current system (x).

If we examine the equation more closely, we can notice a few things:

- The left hand side is a sum of two terms that are positive constants times something squared, so each term is non-negative
- i.e. $\frac{1}{2}m(x')^2 \geq 0$, and $\frac{1}{4}kx^4 \geq 0$
- Each of these terms must also be $\leq C$, otherwise our equation would say:

  (something larger than C) + (something larger than 0) $= C$, which is impossible.
- i.e. we also have upper bounds: $\frac{1}{2}m(x')^2 \leq C$ and $\frac{1}{4}kx^4 \leq C$.
- This can also be understood as saying that we can't have more kinetic (or potential) energy in the system at any point then we had total energy in the system at the start.

- These upper bounds are quite useful in certain engineering applications: even without knowing the exact solution $x(t)$, we can guarantee that it's displacement x and velocity x' stay within a certain range that we know ahead of time. This can be quite useful if we want to guarantee that the system stays within some safe range of positions/velocities.

- This equation also tells us that if $|x|$ is decreasing (i.e. the mass is moving closer towards the origin), then $|x'|$ is increasing, (i.e. its speed is increasing) and vice versa. This helps us develop a qualitative idea of how solutions behave.

This particular example might seem somewhat contrived (I don't know whether or not x^3 springs actually exist), but it helps demonstrate how differential equations are useful because they give a relationship between a function (in this case x) and its derivatives (in this case just $\frac{dx}{dt}$). Even if we don't know how to solve them exactly, they can still be useful in helping us develop a qualitative idea of how solutions behave. We'll learn a lot more about this subject when we cover section 2.5, which deals with autonomous ODEs.